



# Almost diagonal random matrices

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Yad Hashmona, March 30, 2009

# Definition

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- Gaussian, **non-invariant** ensemble of RM

$$\langle H_{nm} \rangle = 0 \quad , \quad \langle |H_{nm}|^2 \rangle = \delta_{nm} + (1 - \delta_{nm})F(|n - m|)$$

$$F(|n - m|) \ll 1$$

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# Important examples

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- Rosenzweig-Porter

Ensemble:  $F(|n-m|) = b \ll 1$

- Power-law banded

RM:

$$F(|n-m|) = \left( \frac{b}{|n-m|} \right)^{2\lambda}$$

$$b \ll 1$$

Mirlin, Fyodorov,  
1996

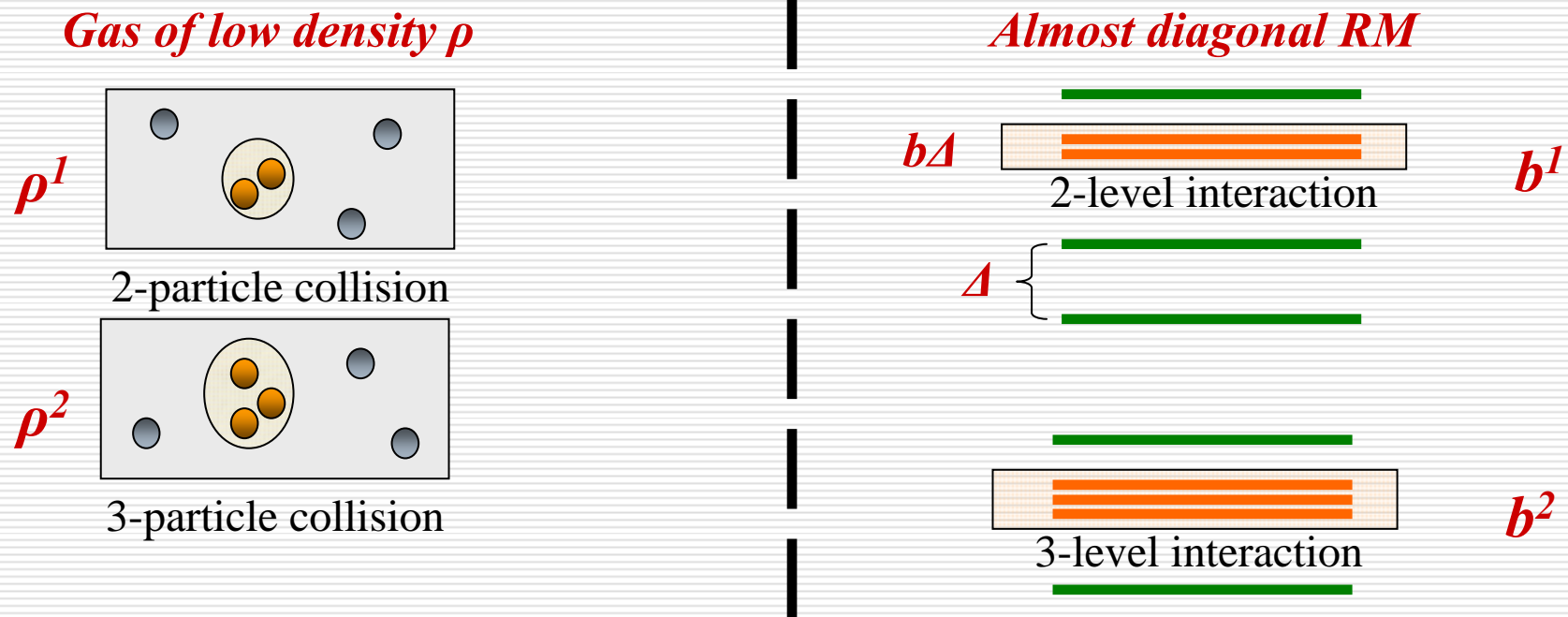
Relevant for describing  
Anderson transition

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# The main idea (Anderson 1958, Levitov 1990)

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- Virial expansion in the number of resonant states (locator expansion)



# Delocalization out of localization

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## □ Return probability

$$P_E(t) = (1/\rho(E)) \sum_{n,m} \langle |\Psi_n(r)|^2 |\Psi_m(r)|^2 \exp[i(E_n - E_m)t] \delta(E_n - E) \rangle$$

Diagonal matrix:

$$P_E(t) = 1$$

Two  
resonant  
levels

Three  
resonant  
levels

**Virial expansion:**

$$P_E(t) = 1 + b P_2(t) + b^2 P_3(t) + \dots$$

**Delocalization:**

$$P_E(t \rightarrow \infty) = 0, \quad (N \rightarrow \infty)$$

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# Critical unitary ensemble of PLBREM

Weak  
delocalization

$$F(|n-m|) = \left( \frac{b}{|n-m|} \right)^2$$

Mirlin, Fyodorov  
1996

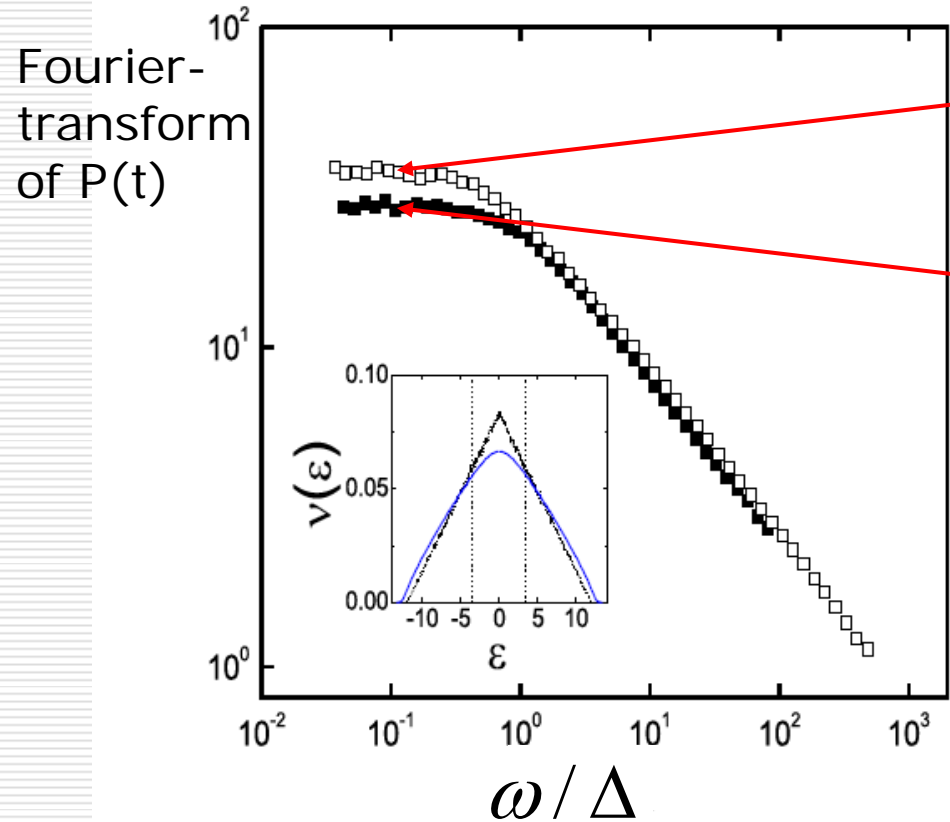
$$P_0(t) = 1 - d_2 \ln(bt) + \frac{d_2^2}{2!} \ln^2(bt) + \dots = \exp[-d_2 \ln(bt)]$$

$$P_0(t) = \frac{1}{(bt)^{d_2}}, \quad d_2 = \frac{\pi b}{\sqrt{2}} \ll 1$$

Delocalization when all  
orders are taken into account

# 3D Anderson model at critical disorder and PLBRM

$$H = \sum \epsilon_i c_i^\dagger c_i + \sum V_{ij} c_i^\dagger c_j + c.c.$$



“□” – critical PLBRM

“■” – 3d Anderson model

(E.Cuevas, VEK, PRB **76**, 235119 (2007))

Fourier-transform of  $P(t)$   
at the mobility edge, one  
single fitting parameter

**$b=0.42$**

# Critical level statistics and PLBRM

Critical level spacing distribution function,  $P_\beta(s,b)$  (Nishigaki, 1999)

Numerics on 3d Anderson model,  $P_\beta(s)$  (Zharekeshev & Kramer, Schweitzer)

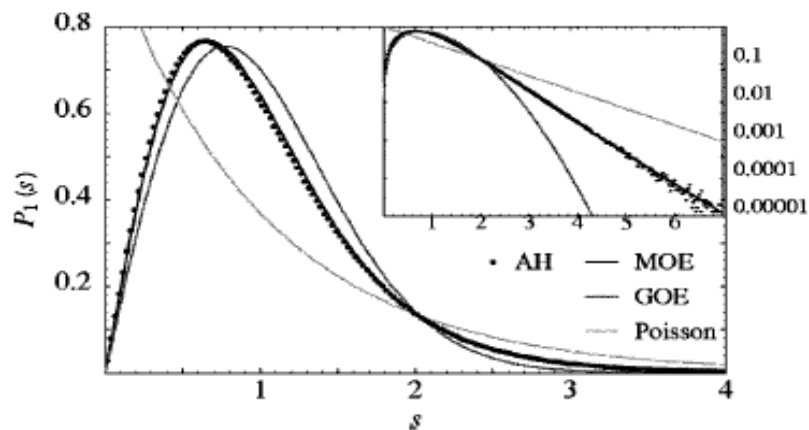


FIG. 5. The LSDF's  $P_1(s)$  of the multifractal orthogonal ensemble (MOE) at  $a=2.95$  and of the Anderson Hamiltonian (1) at the MIT point  $W=16.4$ , on a lattice of size  $L^3=12^3$ . Numerical data are reprinted from Fig. 1 in Ref. [23] courtesy of Zharekeshev.

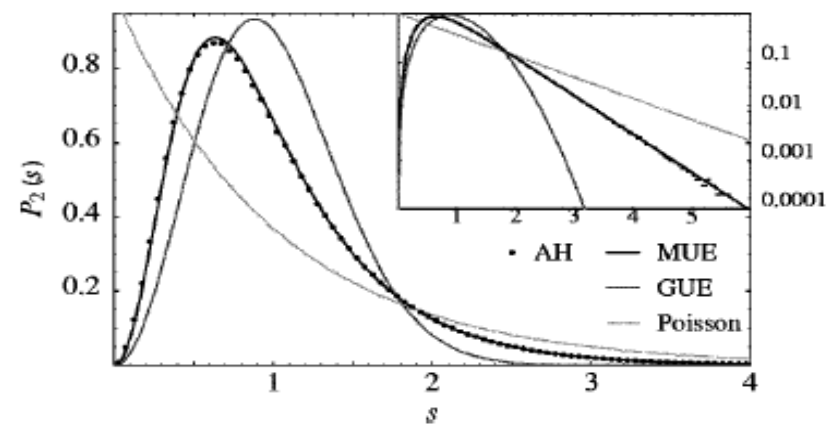


FIG. 6. The LSDF's  $P_2(s)$  of the multifractal unitary ensemble (MUE) at  $a=3.55$  and of the AH (5) under a magnetic field  $\alpha=1/5$  at the MIT point  $W=18.1$ , on a lattice of size  $L^3=5^3$ . Numerical data are reprinted from Fig. 1 in Ref. [24] courtesy of L. Schweitzer.

Potential disorder,  
 $b=0.42$

Potential disorder & magnetic field,  
 $b=0.22$

# How to do the virial expansion?

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- ❑ Trotter formula and combinatorics (O.Yevtushenko and V.E.K. J.Phys.A **36**, 8265 (2003))
- ❑ Super-vector field theory (O.Yevtushenko and A.Ossipov J.Phys.A **40**,4691(2007))

## Application to PLBRM:

O.Yevtushenko and V.E.K. Phys.Rev.E **69**, 026104 (2004)

V.E.K., O.Yevtushenko and E.Cuevas, J.Phys.A **39**, 2021 (2006)

E.Cuevas, VEK, PRB **76**, 235119 (2007)

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# Super-vector field theory

$$\psi = \begin{pmatrix} \varphi \\ \mu \end{pmatrix}$$

→ Bosons (commuting variables)

→ Fermions (anticommuting variables)

$$Q_j = \psi_j \otimes \overline{\psi_j}$$

$$\tilde{P}_E(\omega) \equiv \int P_E(t) e^{i\omega t} dt = \int \mathcal{D}\{Q\} A(Q) \prod_{\alpha=1}^N e^{S_0[Q_\alpha]} \prod_{n \neq m}^N e^{S_p[Q_n, Q_m]}$$

- One-matrix part of the action

$$S_0[Q_\alpha] = \text{Str} \left\{ -a Q_\alpha^2 + i \left( E + \frac{\omega + i0}{2} \Lambda \right) Q_\alpha \right\}$$

$$\Lambda = \text{diag}(1, -1)_{RA}$$

$$a = \frac{1}{2} \langle H_{aa}^2 \rangle = \frac{1}{2\beta}$$

- Weak “interaction” of supermatrices

$$S_p[Q_n, Q_m] = -b_{nm} \text{Str} \{ Q_n Q_m \} \quad b_{nm} = \frac{1}{2} \langle |H_{nm}|^2 \rangle = \frac{1}{2} \mathcal{F}(|n-m|) \ll 1$$

## Re-arrangement of the interaction term

$$\prod_{m \neq n} \exp \left[ -\frac{1}{2} \langle |H_{nm}|^2 \rangle \text{Str} [Q_n Q_m] \right] \rightarrow 1 + \overset{\text{F2}}{\sum_{n>m} V_{n,m}^{(2)}} + \overset{\text{F3}}{\sum_{n>m>l} V_{n,m,l}^{(3)}} + \dots$$

$$V_{n,m}^{(2)} = e^{-\langle |H_{nm}|^2 \rangle \text{Str} [Q_n Q_m]} - 1$$

$$V_{n,m,l}^{(3)} = V_{n,m}^{(2)} V_{n,l}^{(2)} V_{m,l}^{(2)} + V_{n,m}^{(2)} V_{n,l}^{(2)} + V_{n,m}^{(2)} V_{m,l}^{(2)} + V_{n,l}^{(2)} V_{m,l}^{(2)}$$

The term  $F_n$  is a sum of terms containing  $n$  different  $Q$  matrices

Functional integration of  $A(Q) F_n(Q) \exp \left[ -\sum_{i=n,m,\dots} S_0(Q_i) \right]$   
over  $Q$  gives the  $n$ -th virial coefficient  $\tilde{P}_n(\omega)$

# Return probability

$$P_2(t) = \sum_{n \neq p}^N \sqrt{2b_{pn}} \left[ \sqrt{2b_{pn}} |t| e^{-2b_{pn}t^2} + \frac{\sqrt{\pi}}{2} \operatorname{erf} \left( \sqrt{2b_{pn}} |t| \right) \right]$$

$$b_{km} \equiv \frac{1}{2} \langle |H_{km}|^2 \rangle$$



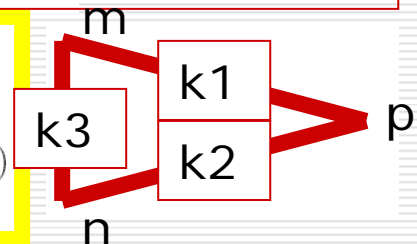
$$P_3(t) = \sum_{\{m, n \neq p\}}^N t^{-2} \sum_{k_1, k_2, k_3=0}^{\infty} (-8b_{pm}t^2)^{k_1} (-8b_{pn}t^2)^{k_2} (-8b_{mn}t^2)^{k_3} \times$$

Sum over the matrix indices

$$\frac{\Xi(k_1, k_2, k_3)}{\Gamma(2[k_1 + k_2 + k_3] - 1)} (k_1 + k_2)(k_1 + k_2 - 1)$$

Sum over the link indices

$$\Xi(k_1, k_2, k_3) = \frac{\Gamma(k_1 - 1/2) \Gamma(k_2 - 1/2) \Gamma(k_3 - 1/2)}{\pi^{1/2} k_1! \pi^{1/2} k_2! \pi^{1/2} k_3!} \times \\ \times (2k_1k_2k_3 - k_1k_2 - k_1k_3 - k_2k_3) \Gamma(k_1 + k_2 + k_3 - 1)$$



# Generality

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Similar expressions for the spectral correlation functions

Valid for any Gaussian almost diagonal RM

Complexity increases with the order of virial coefficient

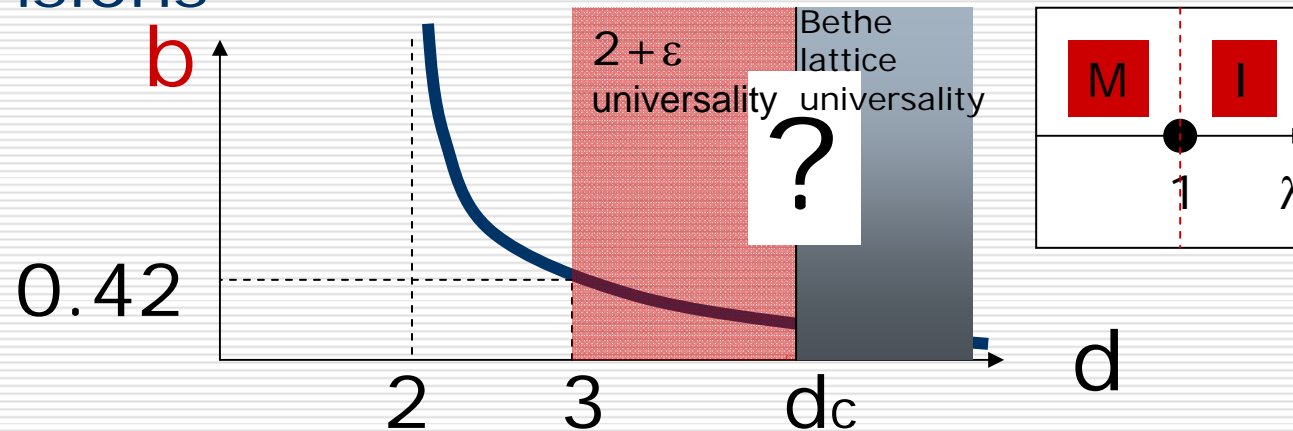
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# Outlook

- Simulating critical and off-critical states at the Anderson transition in higher dimensions

Mirlin, Fyodorov 1996

$$F(|n-m|) = \left( \frac{b}{|n-m|} \right)^{2\lambda}$$



# Conclusion

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- ❑ Expansion in the number of resonant states
  - ❑ Regular method of deriving virial coefficients by the SUSY functional integration
  - ❑ Virial coefficients for any statistics and for arbitrary non-invariant Gaussian RME
  - ❑ Application to localization problem: getting to the transition from the localized side
  - ❑ Properties of Anderson transition in higher dimensions
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