Yad Hashmona ISF research workshop: Random matrices and Integrability

Intersection numbers from duality and replica

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Replica method and supersymmetry

example (1) random magnetic field of Ising model

Parisi-Sourlas, dimensional reduction $d \longrightarrow d-2$, true to all orders, but does not work (Imbrie)

example (2) self avoiding polygons

$n \longrightarrow 0$ limit of O(n) ϕ^4 model, Flory's mean field solution, $\nu = \frac{3}{d+2}$, happens to be exact at d=2, ($\nu = \frac{3}{4}$)

Flory's theory of polymer

$$F = \int_0^T \dot{x}^2(t)dt + g\int dtdt'\delta^d(x(t)-x(t'))$$

$$x(t) \sim t^\nu, \quad \dot{x}(t) \sim t^{\nu-1}$$

$$F \sim t^{2\nu-1} + gt^{2-d\nu}$$

$$\nu = \frac{3}{d+2}\;(\nu = \frac{3}{4})$$

2d branched polymers ⟶ d=0 Yang-Lee singularity (dimensional reduction of Parisi-Sourlas)

$$Z_{Yang-Lee} = \int_c e^{-\frac{1}{g}((x-x_c)\psi+\frac{1}{3}\psi^3}d\psi$$

[4]

self-avoiding polygons (Richard,Guttmann,Jansen)

$$G = \sum_{m,n} p_{m,n} x^m e^{-gn}$$

m: perimeter, n: area x: fugacity g: pressure

$$G(x,g) = g^\theta f(\frac{(x_c - x)}{g^\phi})$$

$$\theta = \frac{1}{3},\ \phi = \frac{2}{3} = \frac{1}{2\nu},\ \nu = \frac{3}{4}$$

scaling function f

$$f = b_0 \frac{d}{ds} \ln A_i(b_1 s)$$

(Brydges-Imbrie,Cardy, Lawler-Shramm-Werner)

[5]

2d multicritical self avoiding polymers (θ-point etc.)

$$F = \int_0^T \dot{x}^2(t)dt + g \int \prod_{j=2}^{k+1} \delta^d(x(t_1) - x(t_j)) \prod_{i=1}^{k+1} dt_i$$

$\nu_{mean} = \frac{k+2}{2+dk}$, $\nu = \frac{k+2}{2(k+1)}$ (d=2 ?).

remark: are these multicritical behaviors exact? not clear since k=2 does not agree with the numerical value θ-point ($\nu_\theta = \frac{4}{7}$).

$$\theta = \frac{1}{k+2}, \ \phi = \frac{k+1}{k+2}, \ \nu = \frac{k+2}{2(k+1)}$$

Exactness (?)for $k > 1$ of Flory's result is due to $\mathcal{N} = 2$ supersymmetry (no correction to mean field result) (Cardy, Saleur)

example (3)

generating function for the intersection numbers of p-spin curves on Riemann surfaces (d=2) is RMT with external source (d=0) at critical point

$$\nu = \frac{k+2}{2(k+1)} = \frac{p}{2(p-1)}, \; p = k+2$$

This exponent ν is same as multicritical self-avoiding polygons.

Witten's general conjecture :

The generating function of the intersection numbers of p-spin curves of moduli spaces satisfies Gelfand-Dikii equation ($p = 2$ case is KdV hierarchy).

$$< \tau_{m_1,j_1} \cdots \tau_{m_n,j_n} >= \frac{1}{p^g} \int_{\bar{M}_{g,n}} c_T(j_1, \cdots, j_n)$$

$$\times \prod_{k=1}^{n} c_1(\mathcal{L}_k)^{m_k}$$

with $(p+1)(2g-2+n) = \Sigma_{i=1}^{n}(pm_i+j_i+1)$.

statement :

$U(s_1, \cdots, s_n) =< \mathrm{tr} e^{s_1 M} \cdots \mathrm{tr} e^{s_n M} >$ **is the generating function of the intersection numbers of** p**-spin curves of moduli spaces, and it satisfies Gelfand-Dikii equation.**

(proof of Witten's general conjecture by RMT)

generalized Kontsevich-Airy matrix model

$$Z = \frac{1}{Z_0} \int dB \exp[\frac{1}{p+1} \mathrm{tr} B^{p+1} - \mathrm{tr} B \Lambda^p]$$

$$Z_0 = \int dB \exp[\sum_{j=0}^{p-1} \mathrm{tr} \frac{1}{2} \Lambda^j B \Lambda^{p-j-1} B]$$

$$F = \sum_{d_{m,j}} < \prod_{m,j} \tau_{m,j}^{d_{m,j}} > \prod_{m,j} \frac{t_{m,j}^{d_{m,j}}}{d_{m,j}!}$$

$$t_{m,j} = (-p)^{\frac{j-p-m(p+2)}{2(p+1)}} \prod_{l=0}^{m-1} (lp+j+1) \mathrm{tr} \frac{1}{\Lambda^{mp+j+1}}$$

(Adler and van Moerbeke (1992))

replica limit $k \to 0$ perturbation for one marked point: we need

$$\lim_{k\to 0} < \mathrm{tr} B^{p_1} \cdots \mathrm{tr} B^{p_n} >$$

B is $k \times k$ Hermitian matrix generating function of above quantity is

$$\lim_{k\to 0} U(s_1, s_2, \cdots, s_n) =< \mathrm{tr} e^{s_1 B} \cdots \mathrm{tr} e^{s_n B} >$$

Theorem:

$$\lim_{k\to 0} U(s_1, \cdots, s_n) = \frac{\lambda}{\sigma^2} \prod_{i=1}^{n} (2\mathrm{sinh}\frac{s_i \sigma}{2\lambda})$$

with $\sigma = \Sigma_{i=1}^{n} s_i$

This theorem provides explicit results of the intersection numbers for arbitrary p and arbitrary genus g for one marked point (one τ).

[13]

$$p = 2; \qquad < \tau_{3g-2} >_g = \frac{1}{(24)^g g!}$$

$$p = 3; \quad < \tau_{(8g-3-j)/3,j} >_g = \frac{1}{(12)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})}$$

with $j = 0$ ($g = 1, 4, 7, ...$) and $j = 1$ ($g = 3, 6, 9, ...$).

$$p = 4; \quad < \tau_{1,0} >_{g=1} = \frac{1}{8}, < \tau_{3,2} >_{g=2} = \frac{9}{8^2 \cdot 5!}$$

derivation of theorem

$$U(s) = \frac{1}{k} < \mathrm{tr} e^{sB} >$$

$$U(s) = \frac{1}{ks} e^{\frac{s^2}{2\lambda}} \oint \frac{du}{2i\pi} e^{\frac{su}{\lambda}} (1 + \frac{s}{u})^k$$

$$\lim_{k \to 0} U(s) = \frac{1}{s} e^{\frac{s^2}{2\lambda}} \oint \frac{du}{2i\pi} e^{\frac{su}{\lambda}} \log(1 + \frac{s}{u})$$

$$= \frac{\sinh(\frac{s^2}{2\lambda})}{(\frac{s^2}{2\lambda})}$$

[15]

$$\lim_{k\to 0} U(s_1,\cdots,s_n) = (-1)^{\frac{n(n-1)}{2}} e^{\sum\frac{s_j^2}{2\lambda}} \oint \prod \frac{du_i}{2i\pi} e^{\sum\frac{u_i s_i}{\lambda}}$$

$$\times \sum_1^n \log(1+\frac{s_i}{u_i})\det\frac{1}{u_i+s_i-u_j}$$

$$= \frac{\lambda}{\sigma^2}\prod_1^n (2\sinh(\frac{s_i\sigma}{2\lambda})$$

[16]

representation by dual matrix M (M is $N \times N$ Hermitian matrix)

N-k duality formula

$$< \prod_{\alpha=1}^{k} \det(\lambda_\alpha - M) >_{A,M} = < \prod_{j=1}^{N} \det(a_j - iB) >_{\Lambda,B}$$

proof is easily obtained by Grassmannian variables

[17]

replica limit $k \to 0$ $\mathbf{(}a_j = 1\mathbf{,}\ \lambda_\alpha = \lambda\mathbf{)}$

$< (\det(\lambda - M))^k >_{A=1,M} = < (\det(1 - iB))^N >_{\Lambda,B}$

r.h.s. in $N \to \infty$ and $k \to 0$ limit becomes Kontsevich model (one marked point),

$$Z = \int dB e^{\frac{Ni}{3}\mathrm{tr}B^3 + iN\mathrm{tr}B\Lambda}$$

l.h.s.

$$\lim_{k \to 0} \frac{1}{k} < e^{k \mathrm{trlog}(\lambda - M)} > = < \mathrm{trlog}(\lambda - M) >_{A=1}$$

[18]

connection of U(s) to Kontsevich model

$$U_{A=1}(s) = \frac{1}{N}\int_{-\infty}^{\infty} d\lambda e^{it\lambda} < \mathrm{tr}\delta(\lambda - M) >_{A=1}$$

$$\frac{1}{Ns}e^{\frac{Ns^2}{2}}\oint(1 - \frac{s}{1-u})^N e^{Nsu}$$

$$\sim \frac{1}{Ns}e^{-Ns}\int \frac{du}{2i\pi}e^{-Nsu^2 - Ns^2u}$$

$$= \frac{\sqrt{\pi}}{(Ns)^{3/2}}e^{-\frac{N}{12}s^3 - Ns}$$

by counting the normalization, it provides

$$< \tau_{3g-2} >_g = \frac{1}{(24)^g g!}$$

p=3 (closing gap $a_j = \pm 1$)

$$U_{A=\pm 1}(s) = \frac{1}{Ns} e^{\frac{Ns^2}{2}} \oint \frac{du}{2i\pi} (1 - \frac{s}{1-u})^{\frac{N}{2}} (1 + \frac{s}{1+u})^{\frac{N}{2}} e^{Nsu}$$

$$= \frac{1}{Ns} \int \frac{du}{2i\pi} e^{-Nsu^3 - \frac{N}{4} s^3 u}$$

$$= \frac{1}{(Ns)^{4/3} 3^{1/3}} A_i(-\frac{N^{2/3}}{4 \cdot 3^{1/3}} s^{8/3})$$

$$A_i(z) = A_i(0)(1 + \frac{1}{3!} z^3 + \frac{1 \cdot 4}{6!} z^6 + \frac{1 \cdot 4 \cdot 7}{9!} z^9 + \cdots)$$

$$+ A_i'(0)(z + \frac{2}{4!} z^4 + \frac{2 \cdot 5}{7!} z^7 + \cdots)$$

This expansion gives the previous result (p=3):[20]

$$< \tau_{(8g-3-j)/3,j} >_g = \frac{1}{(12)^g g!} \frac{\Gamma(\frac{g+1}{3})}{\Gamma(\frac{2-j}{3})}$$

arbitrary p (arXiv:0810.1085)

$$U(s) = \frac{1}{Ns}\int \frac{du}{2i\pi} e^{-\frac{c}{p+1}[(u+\frac{1}{2}s)^{p+1}-(u-\frac{1}{2}s)^{p+1}]}$$

$c = \frac{N}{p-1}\sum \frac{1}{a_\alpha^{p+1}}$

$$< \tau_{1,0} >_{g=1} = \frac{p-1}{24}.$$

$$< \tau_{n,j} >_{g=2} = \frac{(p-1)(p-3)(1+2p)}{p \cdot 5! \cdot 4^2 \cdot 3} \frac{\Gamma(1-\frac{3}{p})}{\Gamma(1-\frac{1+j}{p})}.$$

with $(p+1)(2g-1) = pn + j + 1$

$p \to -1$ limit

$$< \tau >_{g=1} = \frac{p-1}{24} \to -\frac{1}{12}$$

$$< \tau >_{g=2} \to -\frac{1}{120}, \quad < \tau >_{g=3} \to -\frac{1}{252}$$

$$\chi(M_{g,1}) = \zeta(1-2g) = -\frac{B_{2g}}{2g}$$

$\chi(M_{g,1}):$ Euler characteristics, B_{2g}: Bernoulli number,$B_2 = \frac{1}{6}$ etc.

[23]

two-point $U(s_1, s_2)$ p=2:

$$U(s_1, s_2) = \frac{N}{s_1 + s_2} e^{\frac{1}{24N^2}(s_1+s_2)^3}$$

$$\times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(2m+1)} (\frac{s_1 s_2 (s_1 + s_2)}{8N^2})^m \sqrt{s_1 s_2}.$$

$$U(s_1, s_2) = \sum_{n_1, n_2} < \tau_{n_1,0} \tau_{n_2,0} >_g \frac{s_1^{n_1} s_2^{n_2}}{N^{2g}}$$

$$n_1 + n_2 = 3g - 2$$

[24]

p=3:

$$U(s_1, s_2) = \frac{2N'}{s_1 + s_2}(\frac{1}{\sqrt{s_2}})^{\frac{1}{3}} \int_0^\infty dx \mathrm{sh}(\frac{x}{2N'} s_1^{\frac{1}{3}}(s_1+s_2))$$

$$\times A_i(x) A_i(-x(\frac{s_1}{s_2})^{\frac{1}{3}})$$

$$< \tau_{0,0}\tau_{2,0} >_{g=1} = \frac{1}{12}, \quad < \tau_{1,0}^2 >_{g=1} = \frac{1}{12}$$

[25]

comparison with string equation

$$\frac{\partial F}{\partial t_{0,0}} = \frac{1}{2}\sum_{m,m'=0}^{p-2} \eta^{mm'} t_{0,m} t_{0,m'} + \sum_{n=1}^{\infty}\sum_{m=0}^{p-2} t_{n+1,m}\frac{\partial F}{\partial t_{n,m}}$$

where the metric $\eta^{mm'} = \delta_{m+m',p-2}$.

[26]

chiral ring for primary field (g=0 case)

$$< \tau_{0,q_1}\tau_{0,q_2}\tau_{0,q_3} >_{g=0} = \delta_{q_1+q_2+q_3,p-2}$$

$$F = \Sigma < \tau_{0,q_1}\tau_{0,q_2}\tau_{0,q_3} >_{g=0} t_{0,q_1}t_{0,q_2}t_{0,q_3} + O(t^4)$$

structure constant C_{ijk},

$$C_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \quad , (t_i = t_{0,i-1})$$

construction of superpotential W

$$C_{ij}^{k} = \sum_{m=1}^{p-1} C_{ijm}\eta^{mk}.$$

p=4 case

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{2}t_1 t_2^2 + \frac{1}{4}t_2^2 t_3^2 + \frac{1}{60}t_3^5$$

Witten, Dijkgraaf,Verlinde,Verlinde relation

$$C_{ij}^{m} C_{mkl} = C_{ik}^{m} C_{mjl} \quad (ring\ structure)$$

$$\phi_i \phi_j = \sum_k C_{ij}^{k} \phi_k \quad (mod[W'(x)]),\ \phi_i = -\frac{\partial W}{\partial t_i}$$

conclusion

$U(s_1, \cdots, s_n)$ at the critical point in the large N limit describes the intersection numbers of p-spin curves of moduli space $\mathcal{M}_{g,n}$.

The reason is the replica limit for each curves of marked points and the underlying $\mathcal{N} = 2$ supersymmetry, similar to self-avoinding polygons, equivalent to the Yang-Lee singularity.