Universality class of replica symmetry breaking (RSB)
and ist relation to growth phenomena

scaling with RSB orders (on a pseudo-lattice of RSB orders)

two pseudo-dynamical critical points at T=0

continuous and discrete features in the exact RSB limit

analogies with KPZ growth, directed polymers, and more

effective field theory for spin glasses

R. Oppermann (University of Würzburg, FRG)

scaling and universal behavior of replica symmetry breaking (RSB) in the SK-spin glass

R. Oppermann, M.J. Schmidt, Phys. Rev. E 78, 061124 (2008)

M.J. Schmidt, R.O. Phys.Rev. E 77, 061104 (2008)

R.O., M.J.Schmidt, Phys.Stat.Sol. (c) 4, 1-9 (2007)

present results elaborated with

Manuel J. Schmidt (Würzburg, now Univ.Basel/Switzerland)

earlier work and collaborations with

David Sherrington

Mikhail Kiselev, Bernd Rosenow, Heiko Feldmann, …

approaching high orders of RSB, RSB renorm.-group (decimation of orders):
R.O., D. Sherrington, Phys.Rev.Lett. 95, 197203 (2005)
R.O., M.J. Schmidt, D. Sherrington, Phys.Rev.Lett. 98, 127201 (2007)

first T=0 limit calculations in a selfconsistent scheme without T=0 singularities:
R.O., B. Rosenow, Phys.Rev.Lett. 80, 4767 (1998),
H. Feldmann, R.O., Phys.Rev. B 62, 9030 (2000)

metallic quantum spin glass model (finite range, arbitrary dimensions), renormalization group results and relation with Yang-Lee edge singularity:
S. Sachdev, N. Read, R.O., Phys.Rev. B 52, 10286 (1995)

physical model

$$\mathcal{H} = -\sum_{i<j} J_{ij}\, S_i\, S_j \,,$$

J_{ij} Gaussian distributed random (here : zero mean)

(*Sherrington – Kirkpatrick model : 'SK – model'*)

technical aspects

replica trick $\rightarrow$ replicated spin variable S_i^α, $\alpha = 1, 2, \ldots, n,\ n \rightarrow 0$

Gaussian J – integral $\rightarrow$ 4 – spin interaction,

$\underline{\underline{Q}}$ matrix decoupling – field (Hubbard – Stratonovich)

$< Q_i^{\alpha\beta} > \; = \; < S_i^\alpha\, S_i^\beta >$ matrix order parameter,

local in real space, nonlocal in replica space

Parisi's hierarchical choice of $< Q >$:

exact description of spin glass phase in SK – model

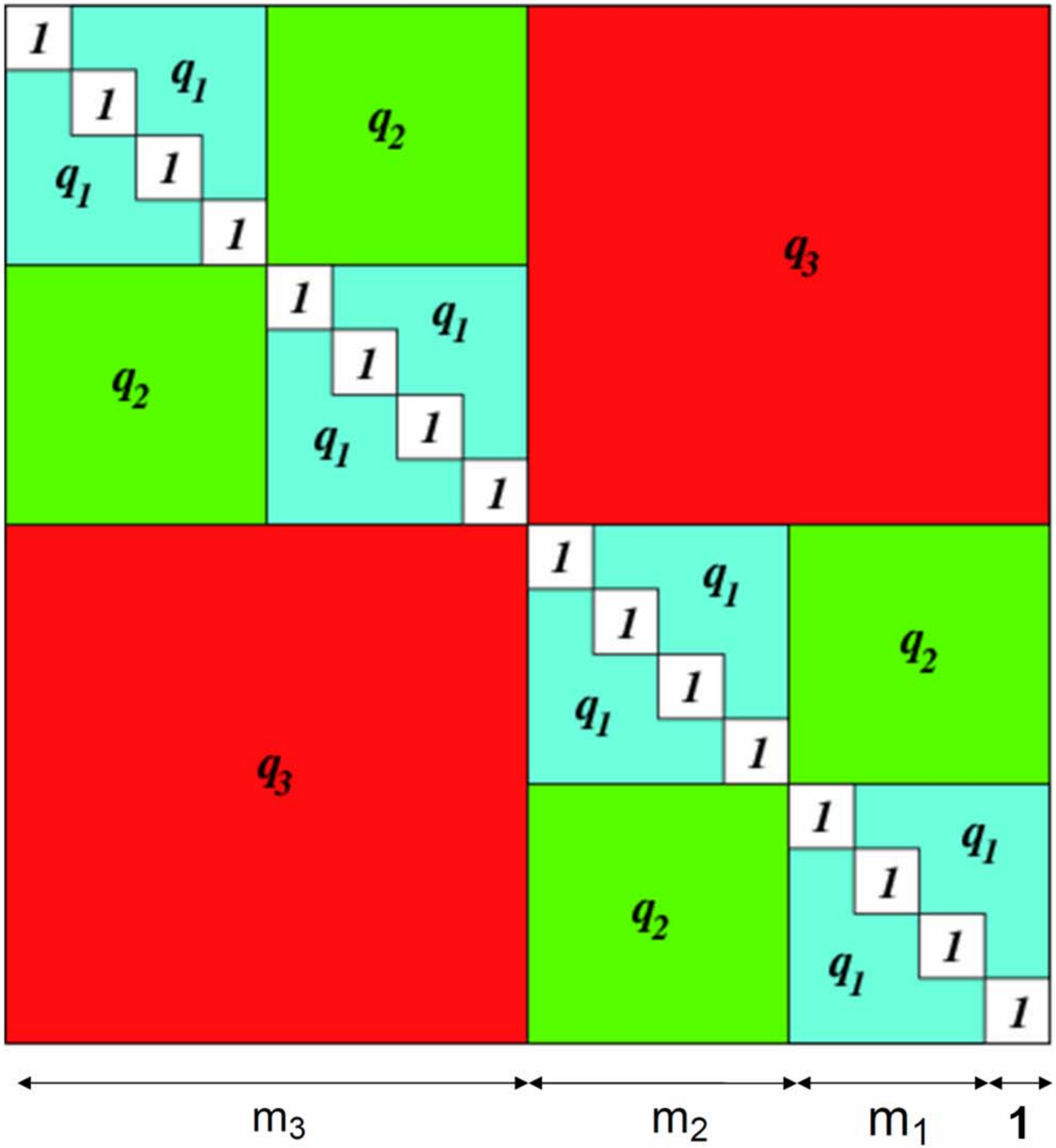
(*proof by Talagrand Ann.Math. 163, 221 (2006)*)

Parisi-type matrix Q

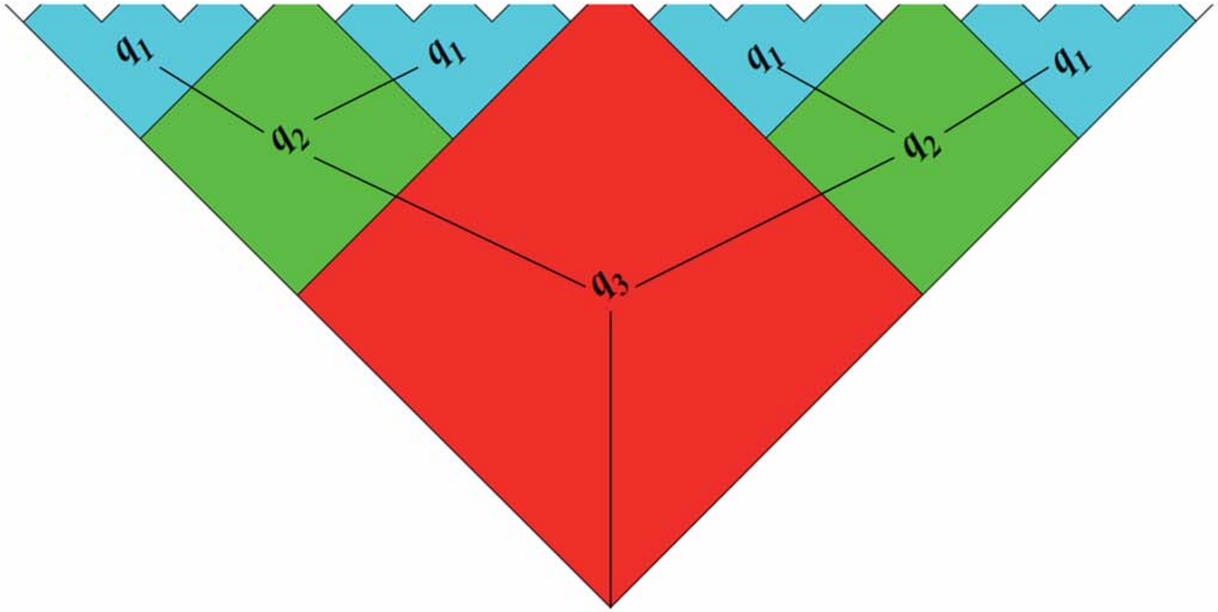
order parameters q
and
box sizes m (or a)

as variational
parameters

$$a_\alpha = \frac{m_\alpha}{T}$$



(choice of parameter labels varies according to technical convenience)



2RSB tree as shown

grows by adding further order parameter levels and the corresponding box sizes

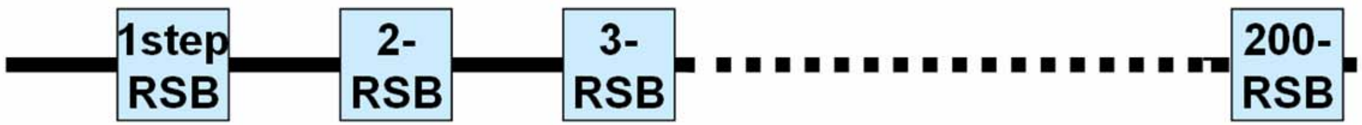
Extremization of the SK ground state energy in arbitrary order of replica symmetry breaking [RSB]

$$E_\kappa = \frac{1}{4}\sum_{\alpha=1}^{\kappa} a_\alpha \left(q_\alpha^2 - q_{\alpha+1}^2\right) - \frac{1}{a_\kappa}\,\mathcal{T}^{\{1\}}_{\kappa+1,\kappa} \log \prod_{\alpha=\kappa}^{1} \mathcal{T}^{\{r_\alpha\}}_{\alpha,\alpha-1}\, e^{|h_1|}$$

$$\mathcal{T}^{\{r_\alpha\}}_{\alpha,\alpha-1} f(h_\alpha) \equiv \int_{-\infty}^{\infty} dh_\alpha \frac{e^{-\frac{1}{2}\frac{(h_{\alpha+1}-h_\alpha)^2}{q_{\alpha+1}-q_\alpha}}}{\sqrt{2\pi(q_{\alpha+1}-q_\alpha)}} f(h_\alpha)^{r_\alpha}$$

in terms of ´transfer matrix´-like operator in RSB space

asymptotic scaling behaviour with RSB orders (one-dimensional)



free energy

$$F_\kappa = -\frac{1}{4} T J^2 \overline{\chi}^2 + \frac{1}{4} \beta J^2 \sum_{i=1}^{\kappa} m_i \left(q_i^{\,2} - q_{i+1}^{\,2} \right) - \delta F_\kappa$$

$$\delta F_\kappa = \frac{T}{m_\kappa} \int_{\kappa+1}^{G} \log \left(\int_{\kappa}^{GE} \ldots \int_{1}^{GE} 2 \cosh (\beta J \, h_{eff}) \right)$$

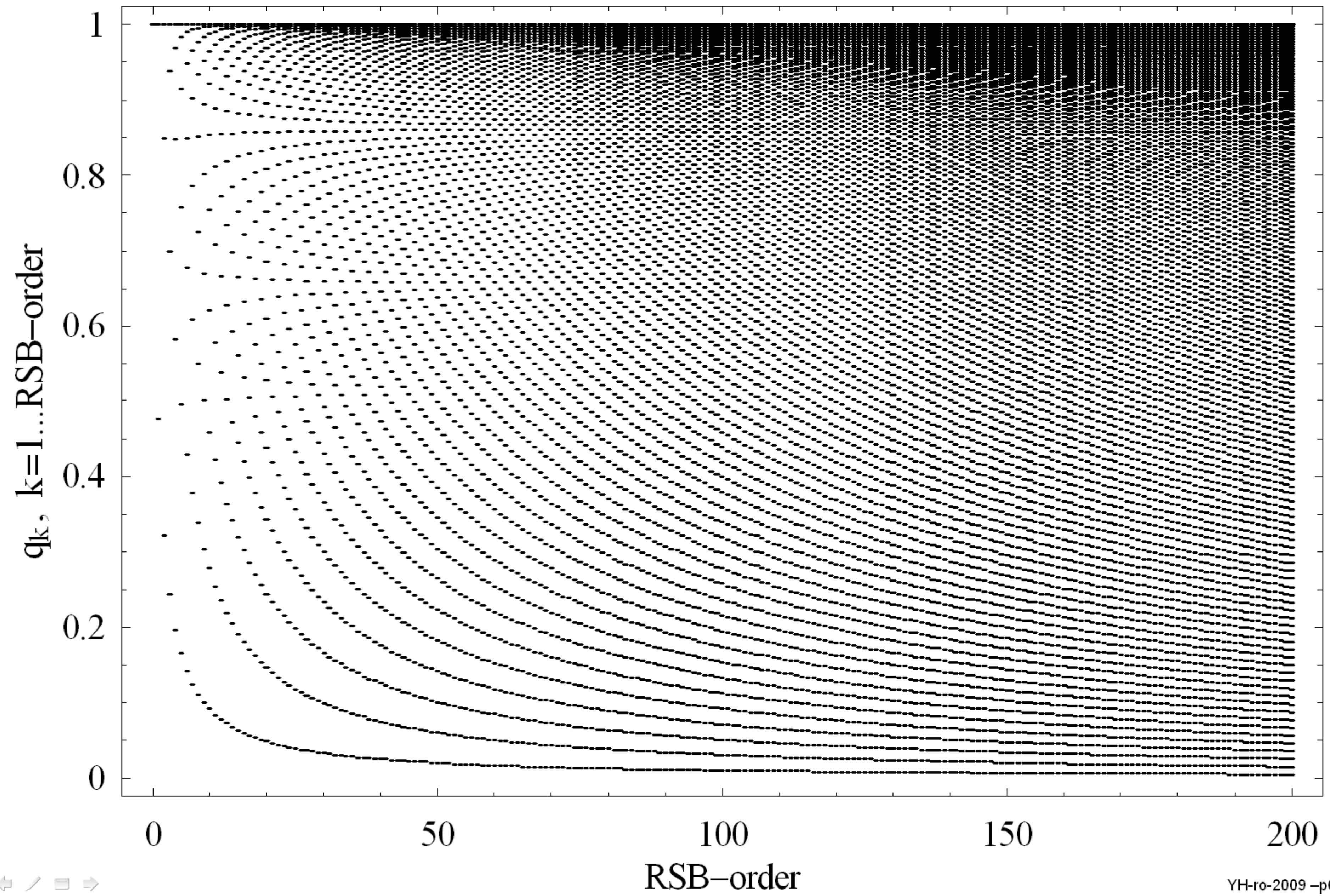
$$\text{defs}: \; h_{eff} = \sum_{\alpha=1}^{\kappa+1} \sqrt{\delta q_\alpha} \, z_\alpha + H , \qquad \delta q_\alpha = q_\alpha - q_{\alpha-1}$$

$$\int_{\alpha}^{GE} Y(h_\alpha) = \int_{-\infty}^{\infty} dh_\alpha \, \frac{e^{\frac{-(h_{\alpha+1} - h_\alpha)^2}{2 \, \delta q_\alpha}}}{\sqrt{2 \pi \, \delta q_\alpha}} \, Y^{r_\alpha} (h_\alpha)$$

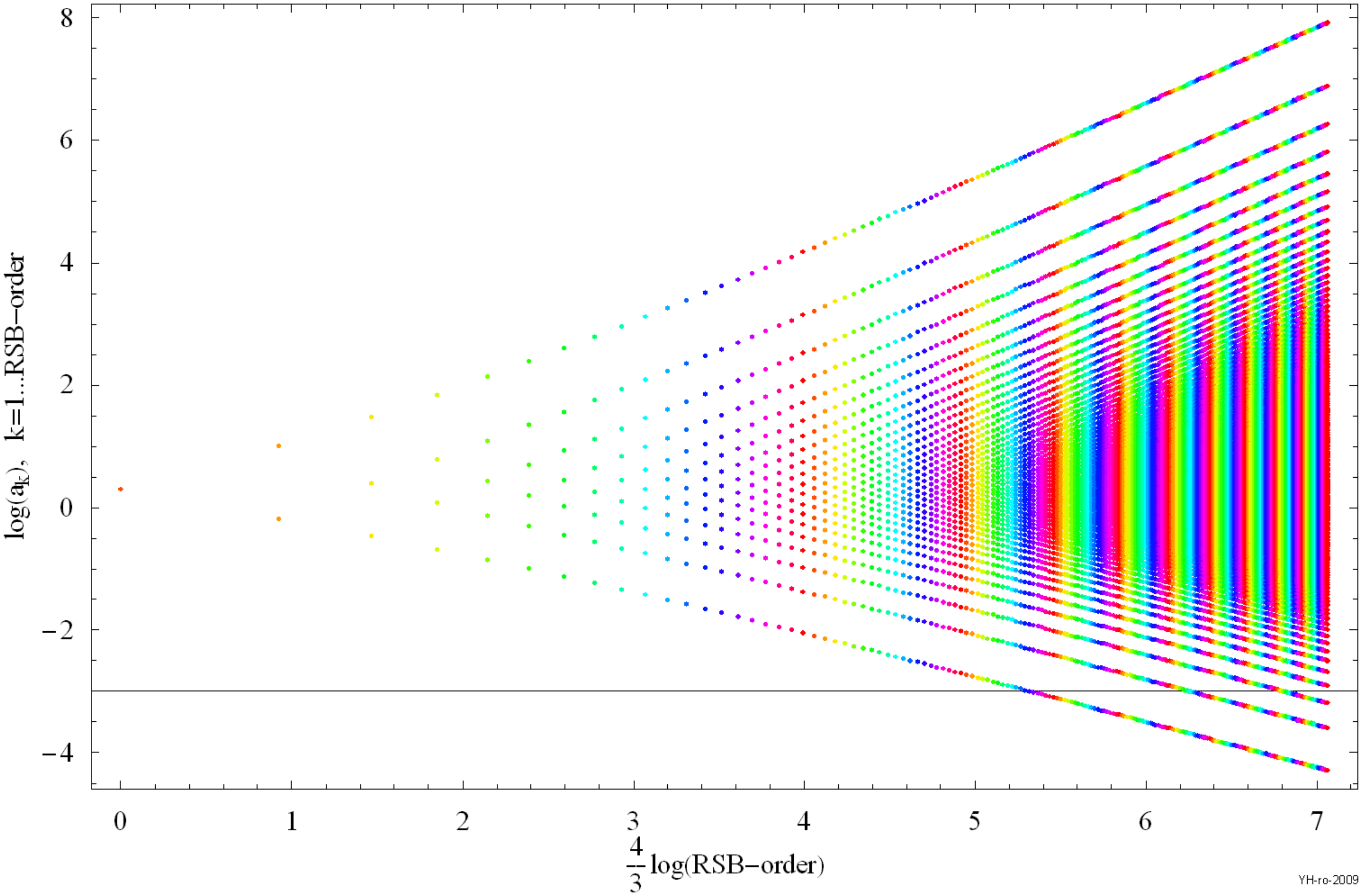
T=0 selfconsistent solutions
for

order parameters q,
box-size levels m,
(and their ratios r)

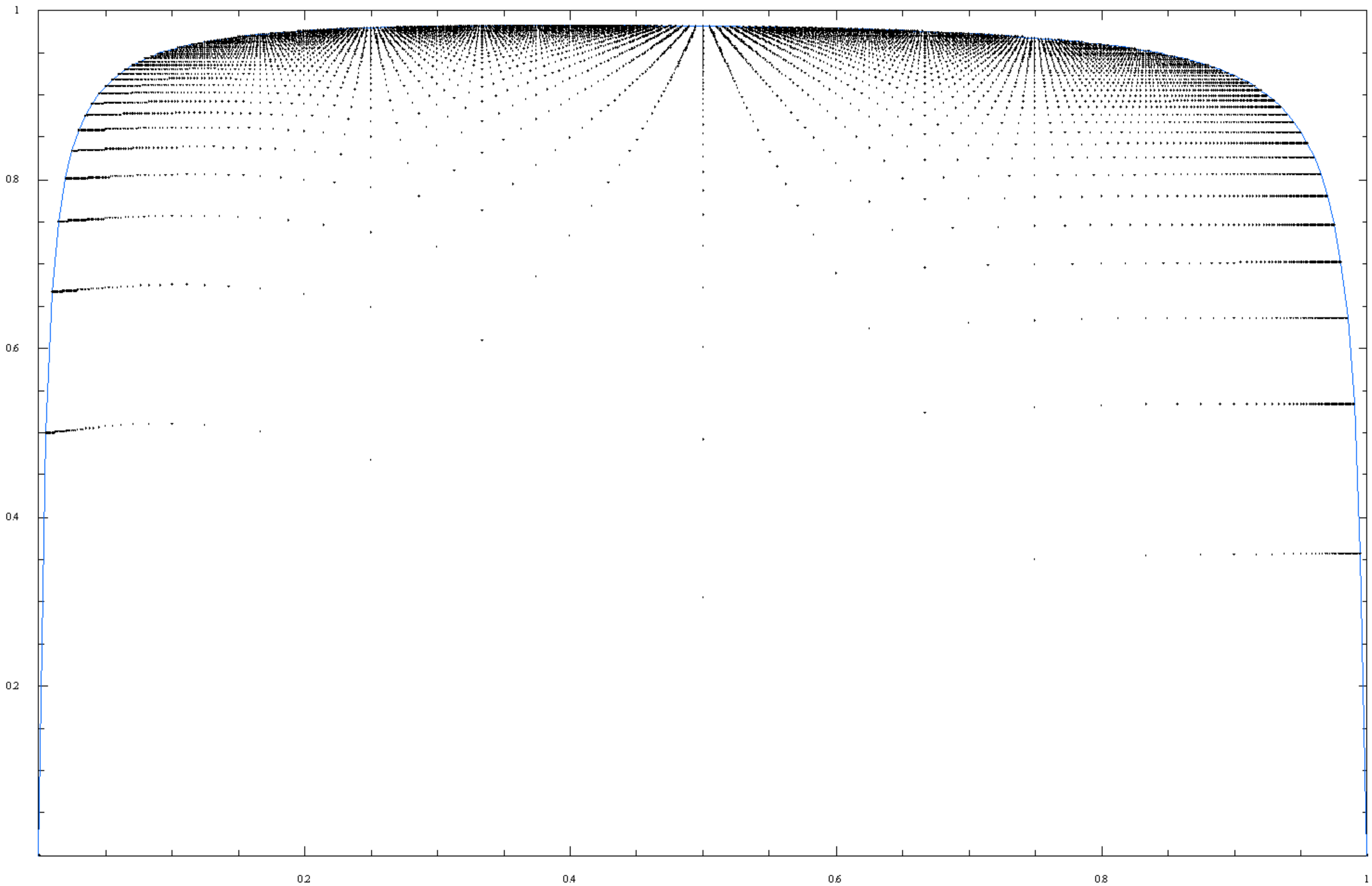
Extremization of T=0-energy: q-levels from 0-RSB to 200-RSB



a-levels from 1-200RSB (log-log plot)
dense regime: continuous spectrum of Parisi block size ratios
below and above: origin of two discrete spectra

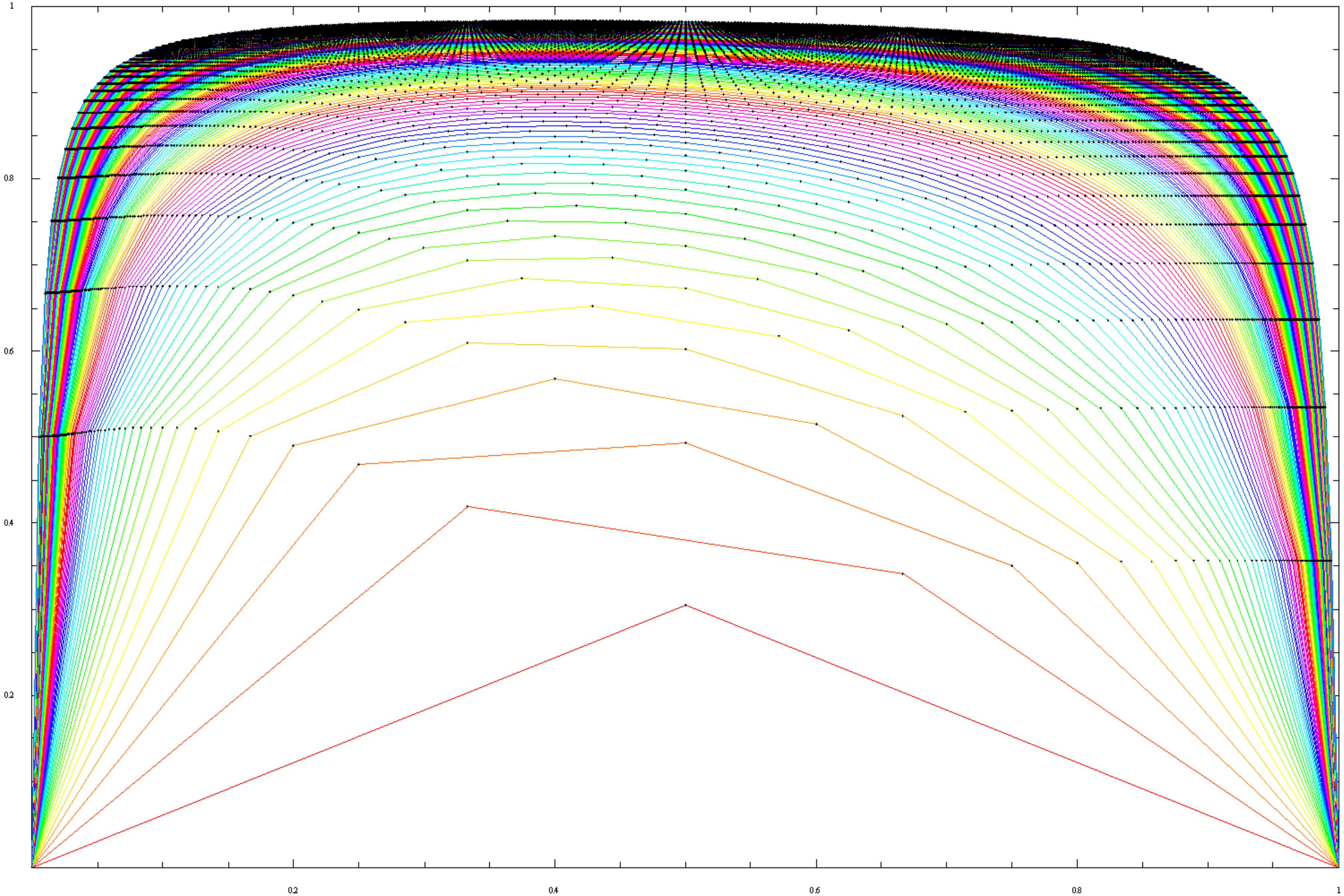


Parisi box size ratios for RSB-orders 2,3,…,200



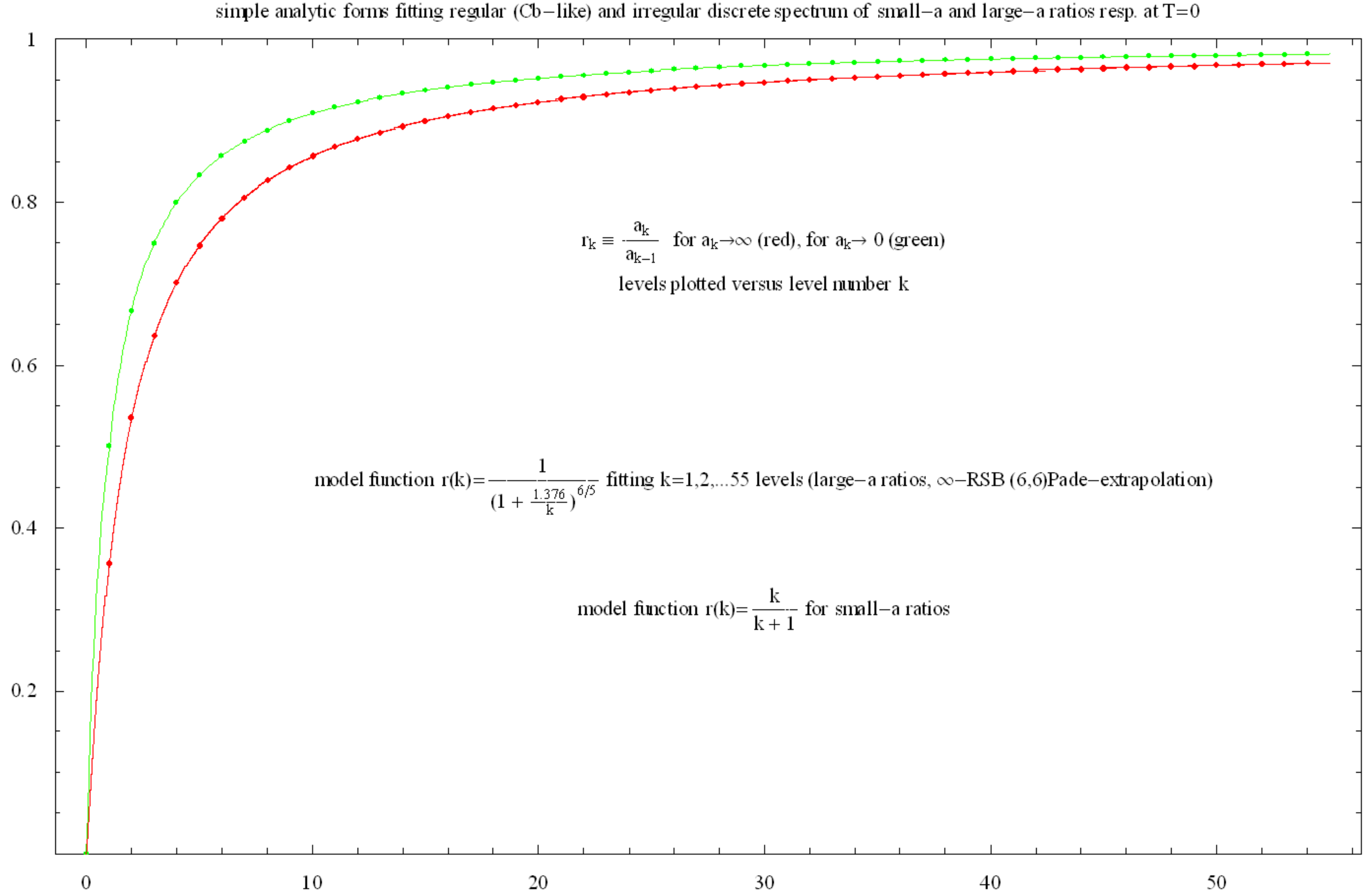
ratios plotted versus level number normalized by RSB-order within interval [0,1], hence at 1/2 for 2RSB, at 1/3 and 2/3 for 3RSB, (1/4,1/2,3/4) for 4-RSB etcetera, see also following slide

Parisi box size ratios on lhs and rhs remain discrete in RSB limit, continuous on top of the box



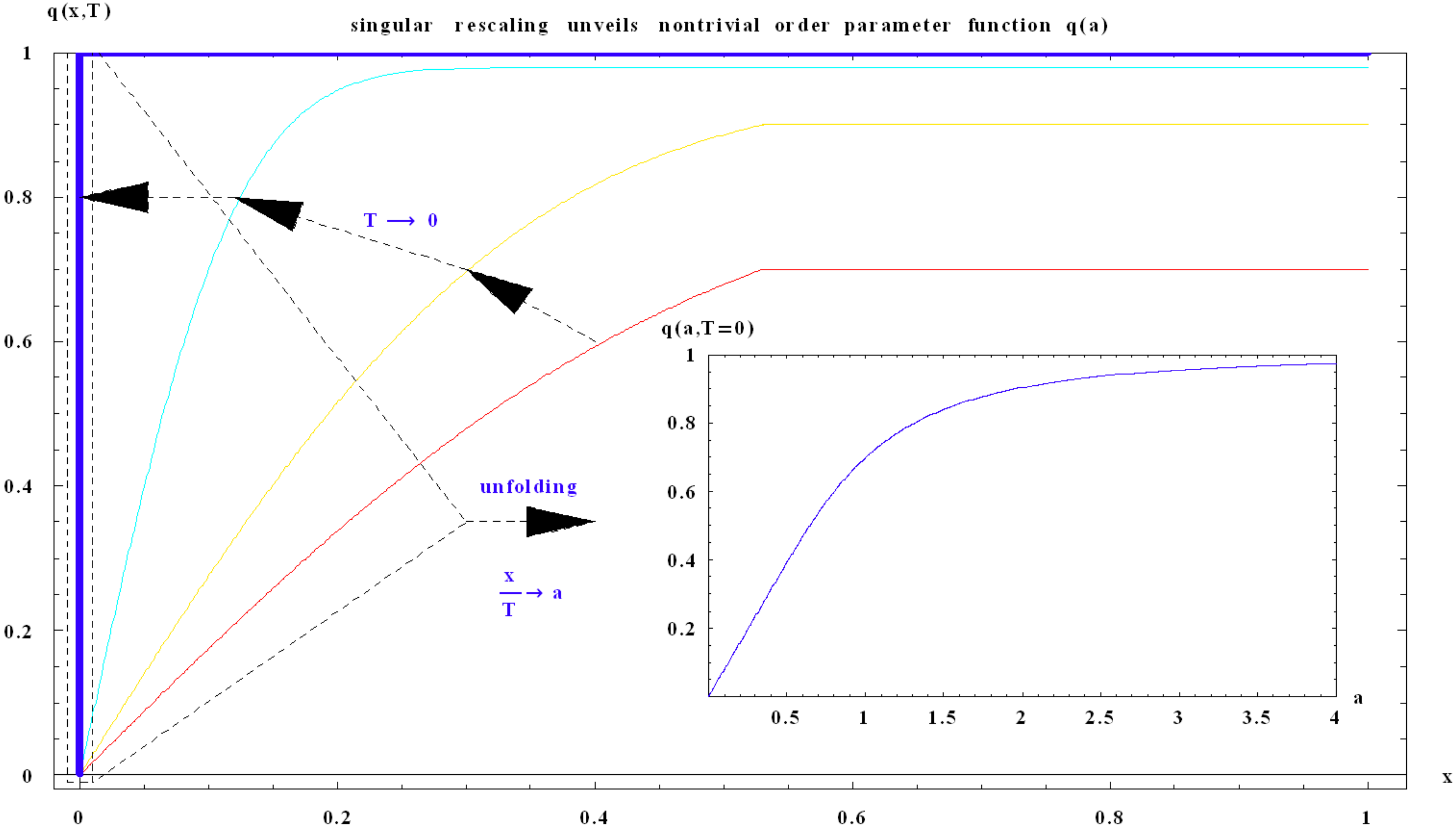
discrete spectra of ratios plotted vs level numbers

red: CP1 thermally unstable, green:CP2, field unstable



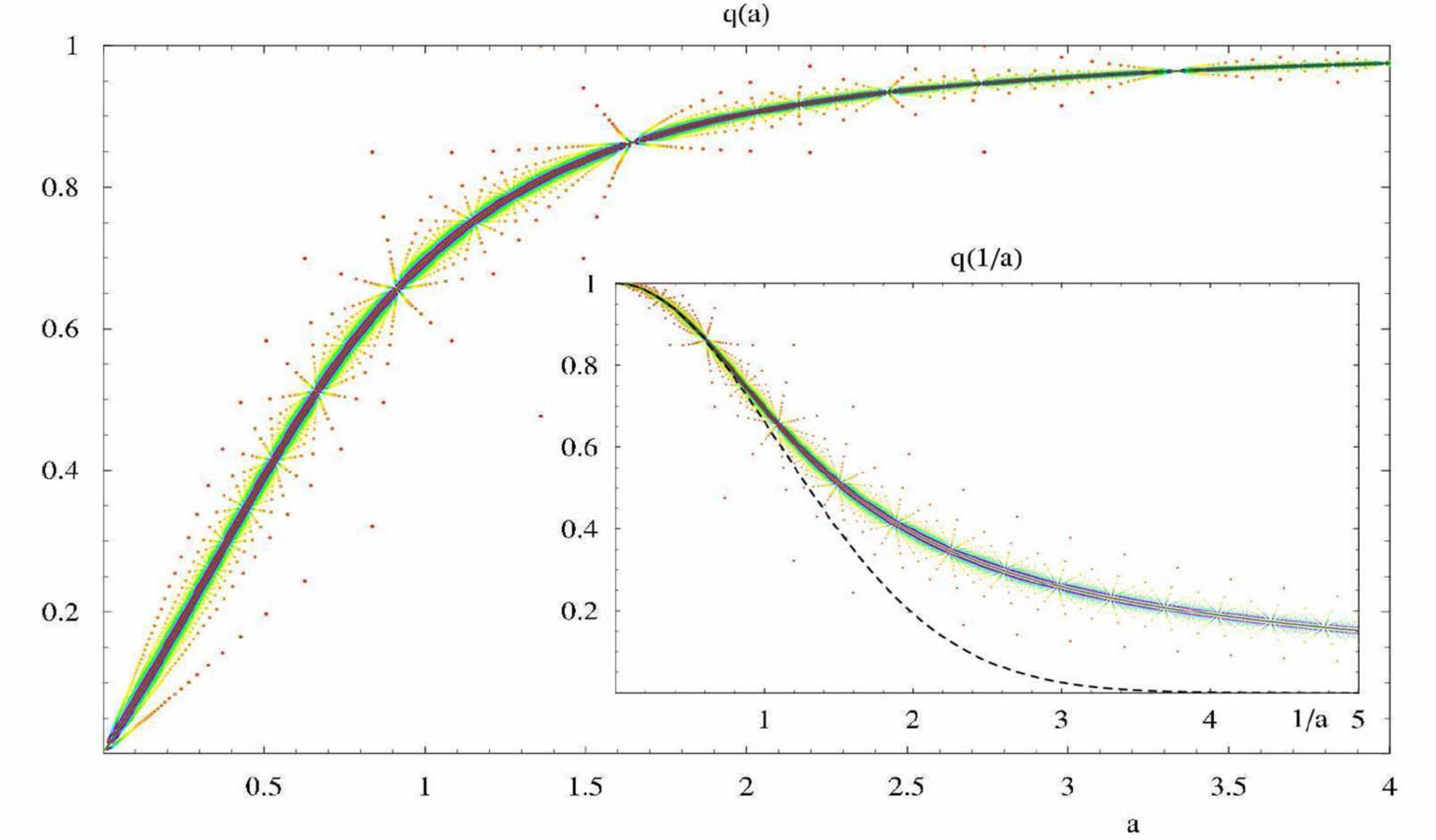
order parameter function depending on Parisi block sizes m(T) → x

singular 1/T-normalization m(T)/T = a(T) → a reveals the low temperature limit



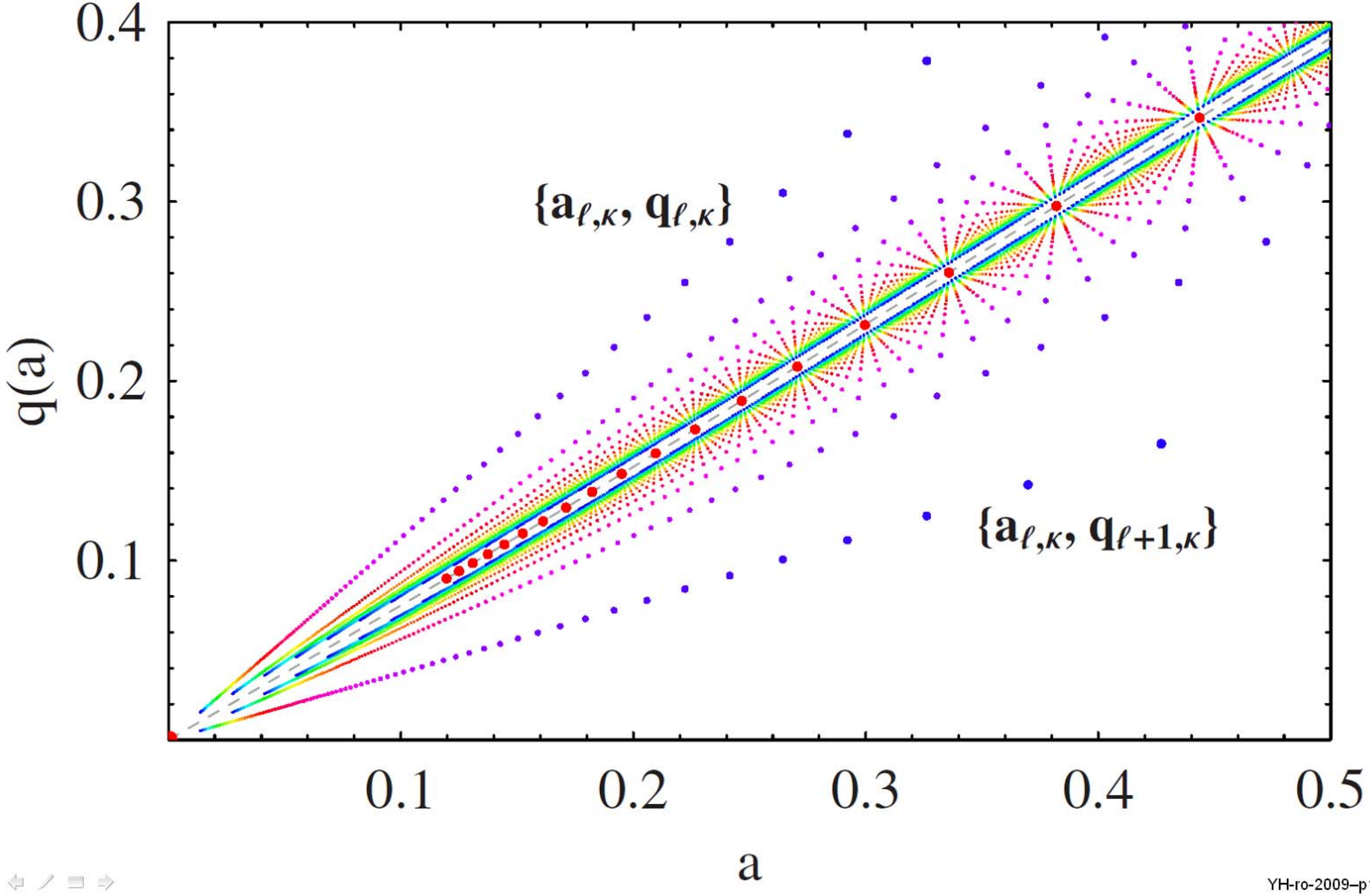
order parameter function q(a) at T=0

as a fixed point function under RSB flow

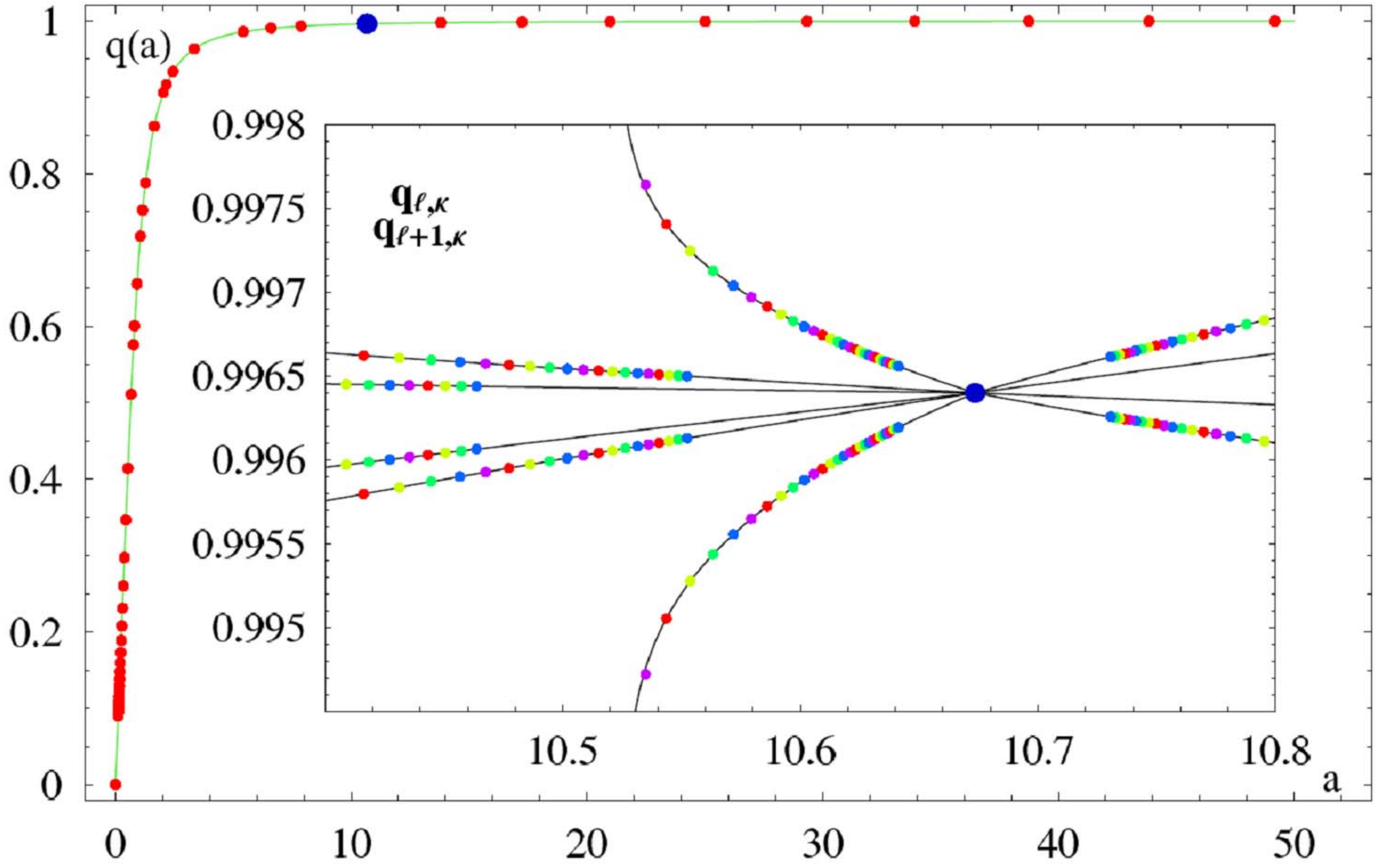


RSB-flow towards fixed points

small-a regime



RSB-fixed point example: large-a regime



analytic model of the order function

1st oversimplified version:

$$q_{\text{model}}(a) = \frac{1}{2\gamma}\sqrt{\frac{\pi}{w}}\, a\, \text{erf}\left(\frac{\sqrt{w}\,\gamma}{\sqrt{a^2+w}}\right)$$

not able to satisfy simultaneously all exact or numerically obtained criteria

equilibrium susceptibility :

$$\chi(T=0) = \int_0^\infty da\, a\, q'(a) = 1$$

ground state energy :

$$E_0 = \int_0^\infty da\left(1-(q(a))^2\right) = -0.763166\ldots$$

slope q'(a) at a = 0 : $q'(0) = 0.74345\ldots$

large – a expansion $q(a) = 1 - \frac{0.411}{a^2} + O\left(\frac{1}{a^4}\right)$

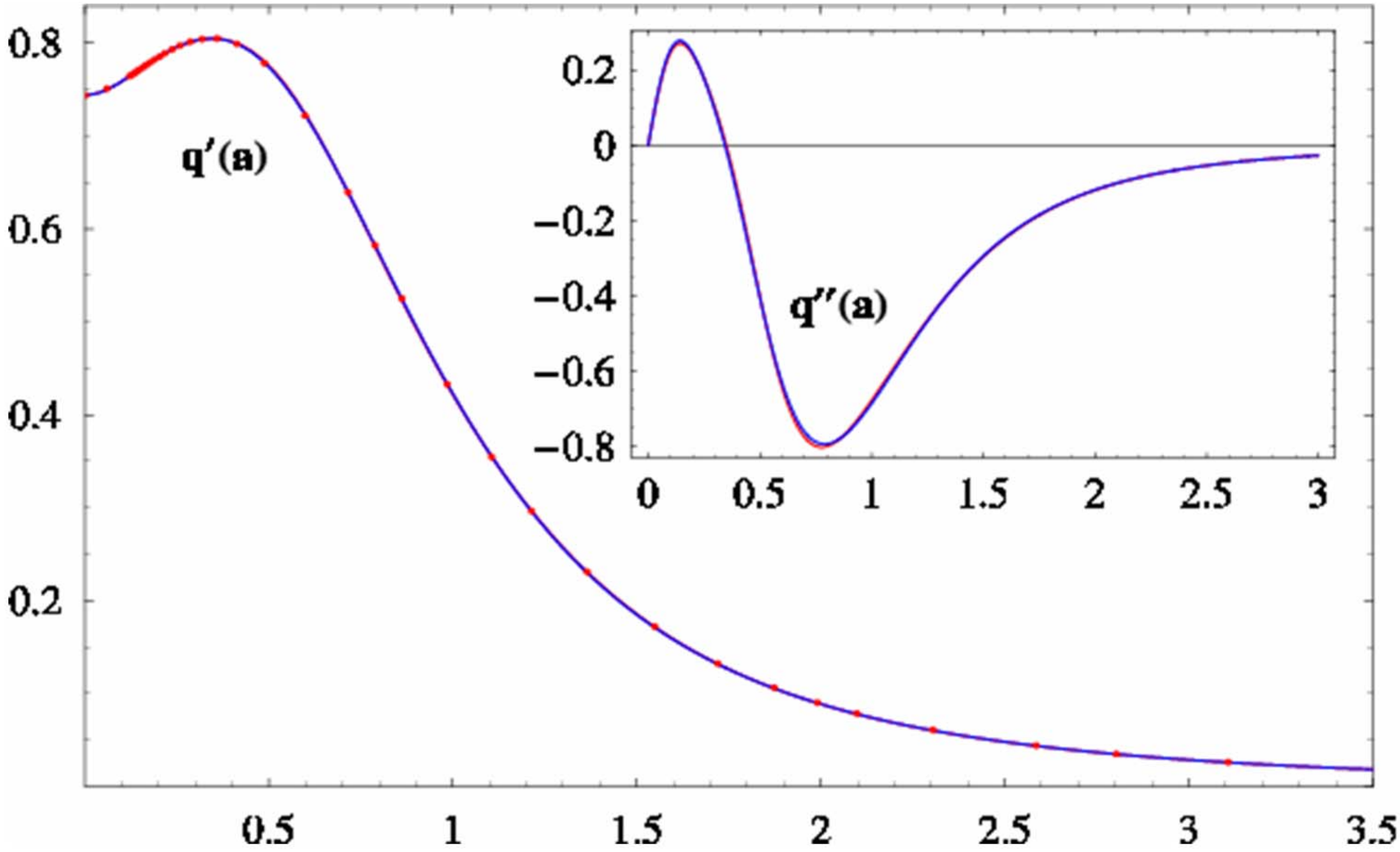
refined model with 5 fit parameters satisfies all criteria

$$q_{model}(a) = \frac{a}{\sqrt{a^2 + w(a)}} \, {}_1F_1\left(\alpha, \gamma, -\frac{\xi^2}{a^2 + w(a)}\right)$$

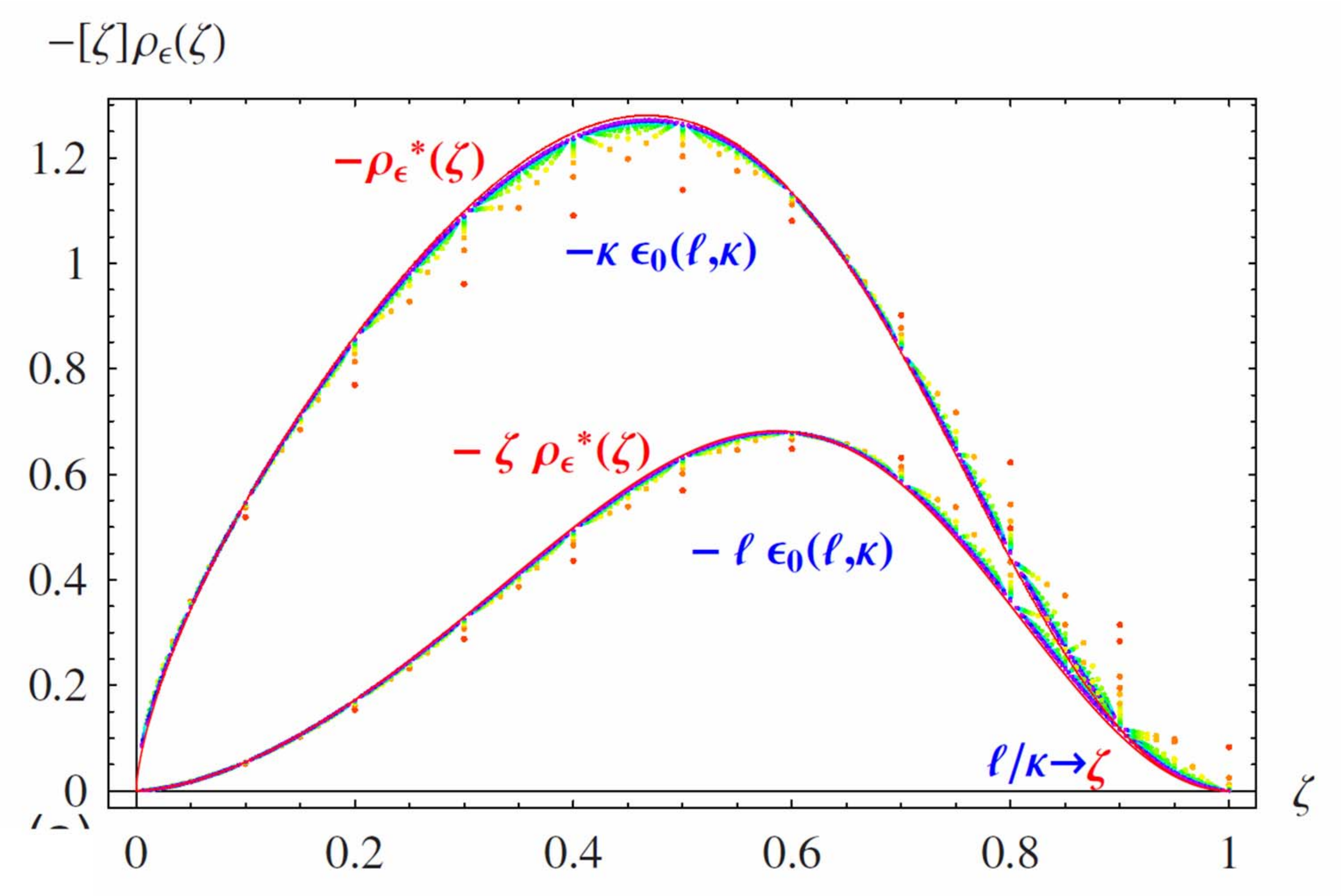
$\alpha \approx 0.558$, $\gamma \approx 1.87$, $\xi^2 \approx 1.41$, and

a small mass term $w(0) \approx 0.067$, $w(\infty) = 0$, eg $w(a) = \frac{w(0)}{1 + c\,a^2}$

nontrivial structure resolved in q´(a) and q´´(a)



energy density: RSB-flow and fixed point function



analytic 1F1-model satisfies a simple 2nd order ODE as shown for the ´mass-free` case w(a)=0:

$$\frac{1}{2}\frac{a^4}{\xi^2}\frac{d^2}{da^2}q(a)+\left(\left(\frac{3}{2}-\gamma\right)\frac{a^2}{\xi^2}-1\right)a\frac{d}{da}q(a)+2\alpha\, q(a)=0$$

boundary conditions $q(0)=0$ and $q(\infty)=1$

physical interpretation by means of a transformation into 1st-order differential equations.

two auxiliary functions

$$p_1(a)=exp\left\{\frac{2}{\xi}\int\left(\left(\frac{3}{2}-\gamma\right)\frac{\xi}{a}-\frac{\xi^3}{a^3}\right)da\right\},p_2(a)=4\alpha\frac{\xi^4}{a^4}p_1(a)$$

and

$$z(a)\equiv p_1(a)q'(a)/q(a)$$

lead to the Riccati equation

$$z'(a)+\frac{1}{p_1(a)}z(a)^2+p_2(a)=0,\quad a\geq 0$$

define a new field

$$\phi(a) \equiv \frac{1}{\sqrt{2\alpha}} \left(\frac{a}{\xi}\right)^{2\gamma-1} e^{-\xi^2/a^2} z(a)$$

which obeys a Langevin-type relaxational equation

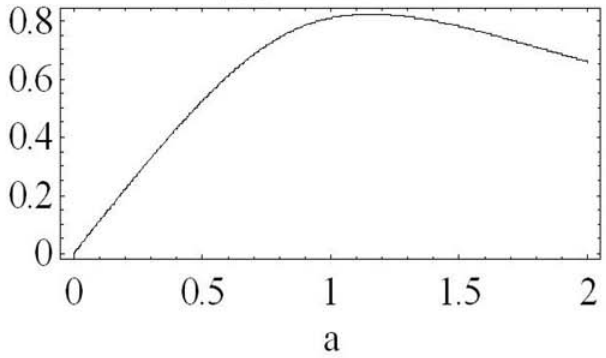
$$\frac{d}{d(\frac{\xi}{a})}\phi(a) = \frac{\delta}{\delta\phi(a)}\mathcal{H}[\phi]$$

$$\mathcal{H}[\phi] = \sqrt{8\alpha}\left(\phi(a) + \frac{1}{6}\phi(a)^3\right) - \left((\gamma - 1/2)\frac{a}{\xi} + \frac{\xi}{a}\right)\phi^2(a)$$

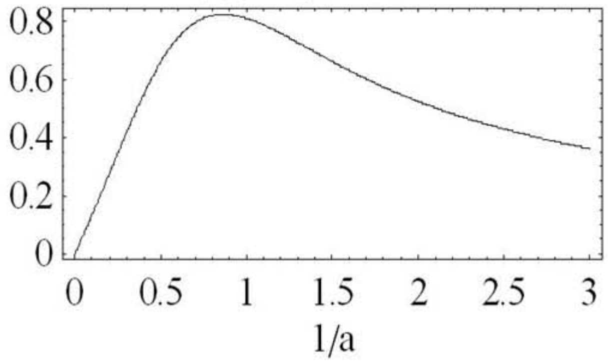
$$\phi(a) = -\frac{1}{\sqrt{2\alpha}}\frac{d}{d\frac{\xi}{a}} log\left(q(a)\right)$$

$\rightarrow$ interpretation of $1/a$ as physical pseudo-time

$\phi(a) \sim a$ for $a \rightarrow 0$



$\phi(a) \sim 1/a$ for $a \rightarrow \infty$



finite low temperatures
and
magnetic field dependence

free energy for arbitrary RSB order (finite T, finite H)

$$F_\kappa = -\frac{1}{4} T J^2 \overline{\chi}^2 + \frac{1}{4}\beta J^2 \sum_{i=1}^{\kappa} m_i \left(q_i^2 - q_{i+1}^2\right) - \delta F_\kappa$$

$$\delta F_\kappa = \frac{T}{m_\kappa} \int_{\kappa+1}^{G} \log \left(\int_{\kappa}^{GE} \dots \int_{1}^{GE} 2\cosh(\beta J\, h_{eff}) \right) := \frac{T}{m_\kappa} \int_{\kappa+1}^{G} \log f_\kappa (h_{\kappa+1})$$

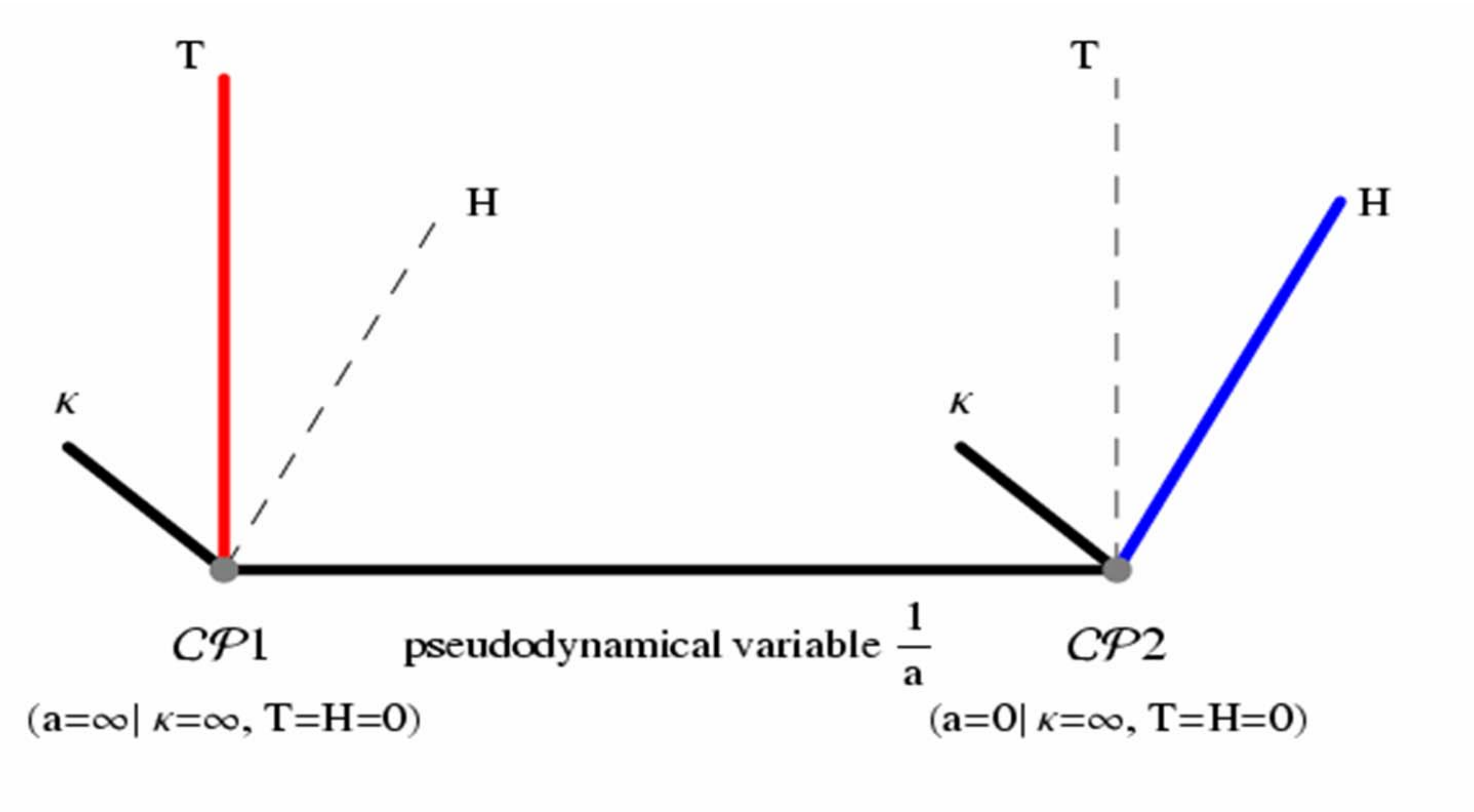
recursive structure : $$f_{\alpha+1}(h_{\alpha+2}) = \int_{\alpha+1}^{GE} f_\alpha (h_{\alpha+1})$$

defs : $$\int_{\alpha}^{GE} Y(h_\alpha) = \int_{-\infty}^{\infty} dh_\alpha \frac{e^{\frac{-(h_{\alpha+1}-h_\alpha)^2}{2\,\delta q_\alpha}}}{\sqrt{2\pi\,\delta q_\alpha}} Y^{\frac{m_\alpha}{m_{\alpha-1}}}(h_\alpha)\ ,$$

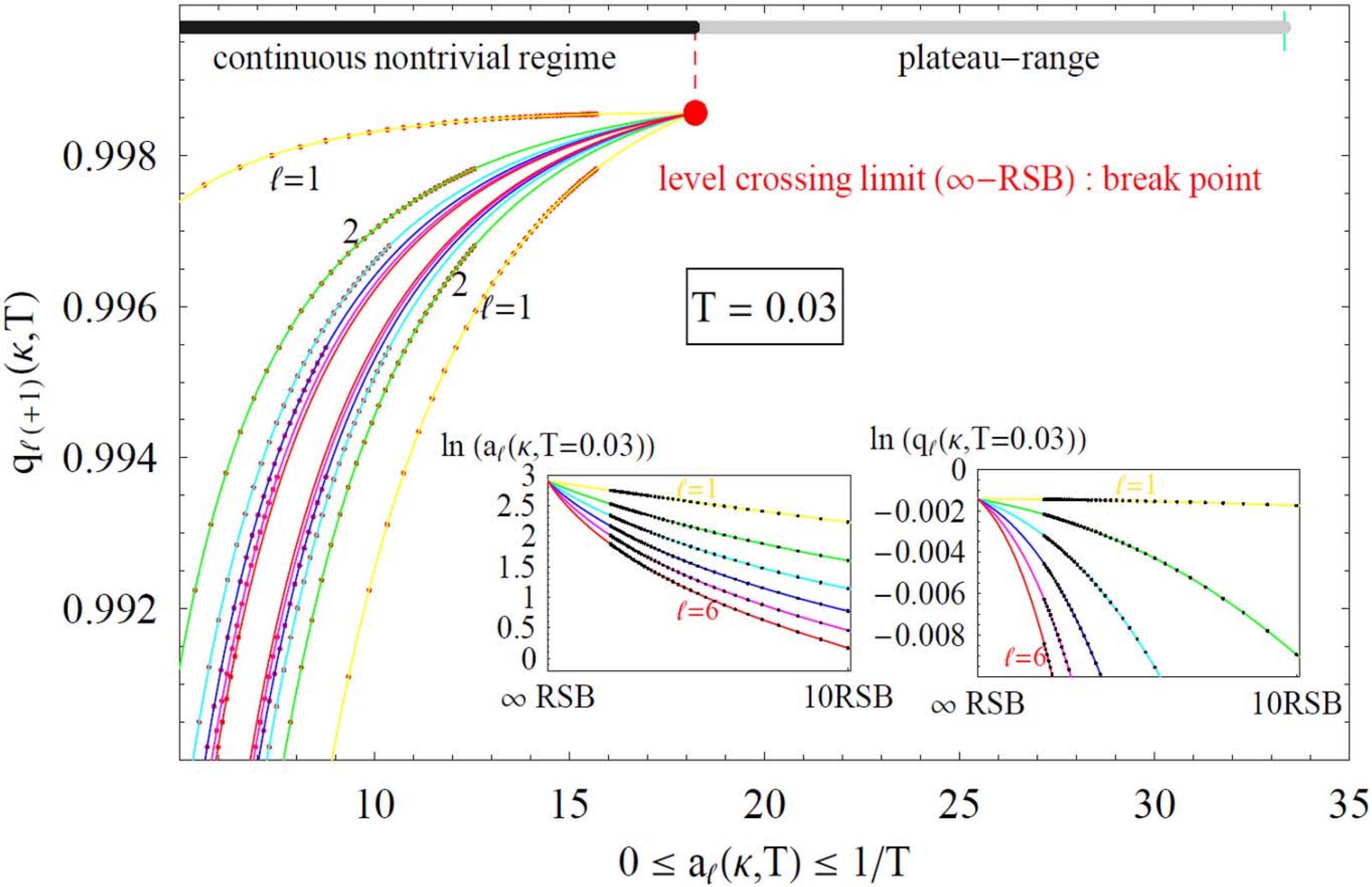
$$h_{eff} = \sum_{\alpha=1}^{\kappa+1} \sqrt{\delta q_\alpha}\, z_\alpha + H\,, \qquad \delta q_\alpha = q_\alpha - q_{\alpha-1}$$

Two critical points

scaling scenario of (SK)-RSB



forbidden level-crossing determines the break point (as shown here for low T=0.03[J])



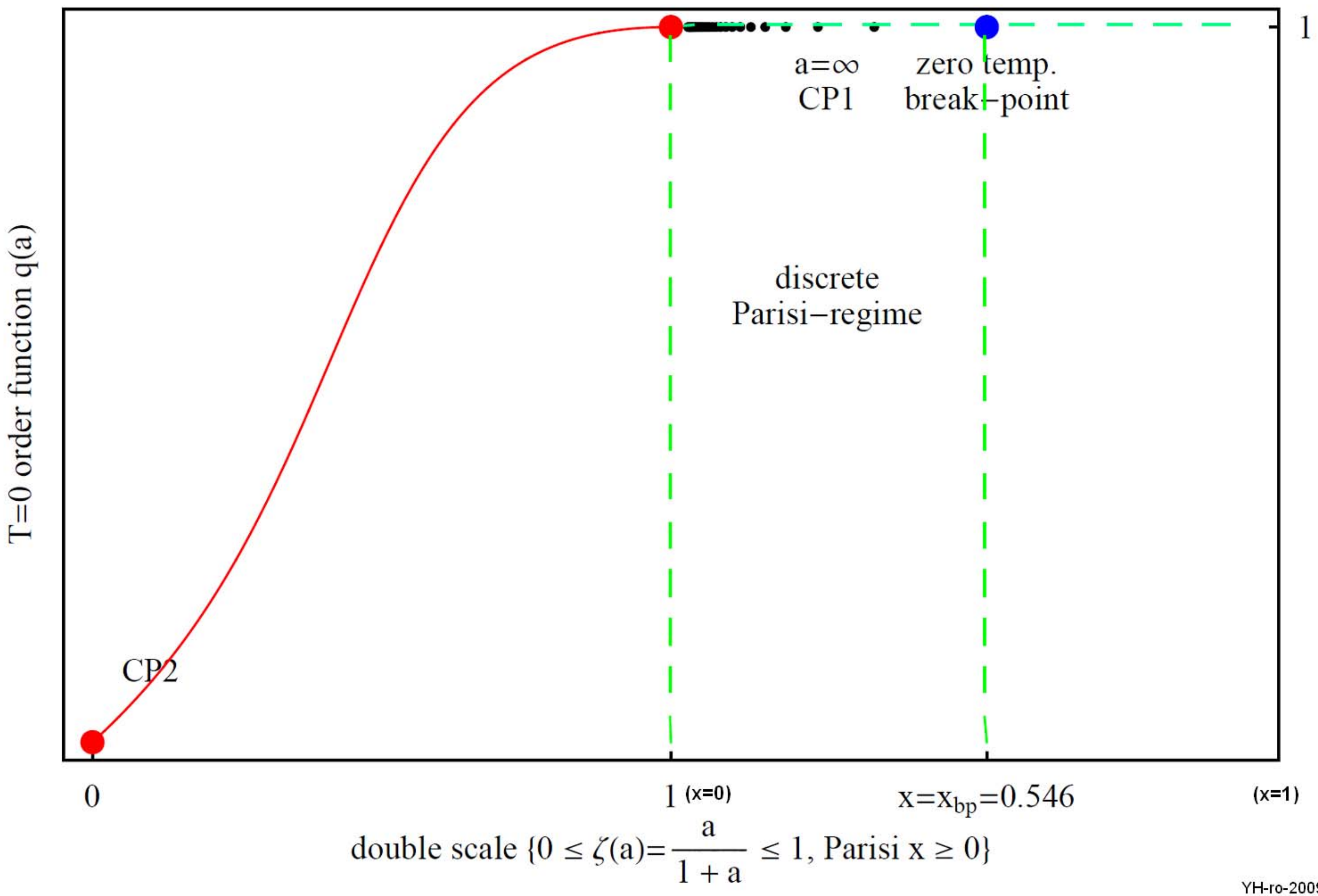
weak temperature dependence of the break point extrapolates to 0.546 for T=0

discrete ratios determine all other discrete values of the Parisi box size parameter

x is not a continuous variable in the ‚continuum limit‘, at least q=1 is realized only at a discrete set of points below the break point!

discrete and continuous regimes are connected at the critical points CP1 and CP2

double scale representation of the order function at T=0
and discreteness of the Parisi sector (x > 0)



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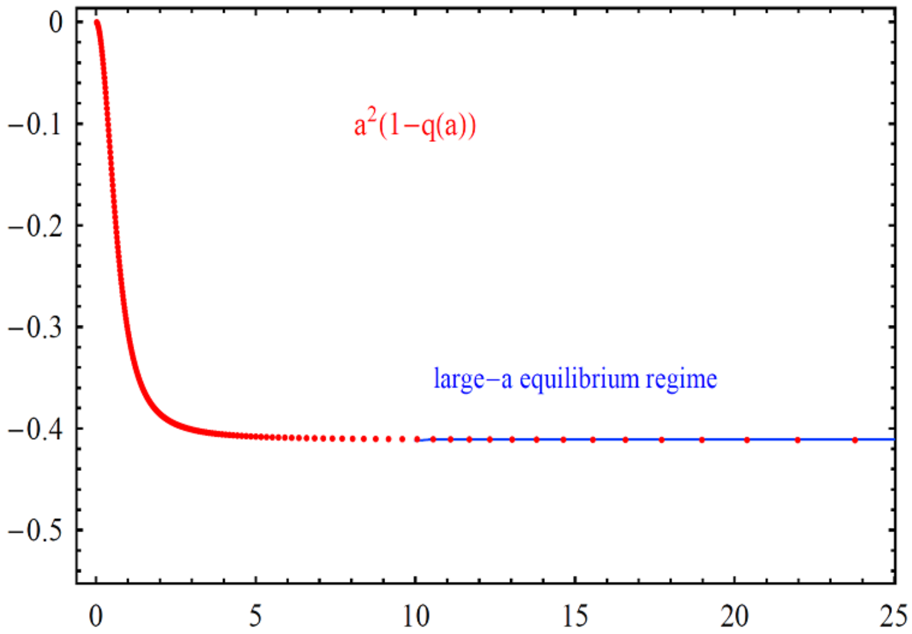
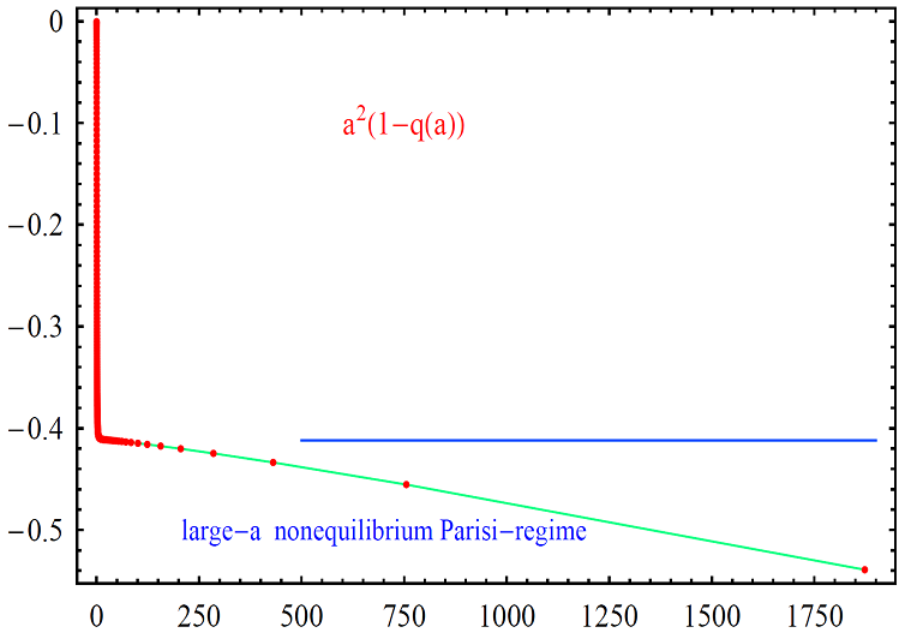
further discrete features of the
infinite RSB limit
(so-called ‚continuum limit‘)

inspection of the large-a expansion
by approaching the RSB limit

asymptotic $q(a\to\infty)$ behavior: control of $\frac{1}{a^2}$ coefficient



$q(a)\approx 1-\frac{0.412}{a^2}$ in continuous regime and breakdown for largest a



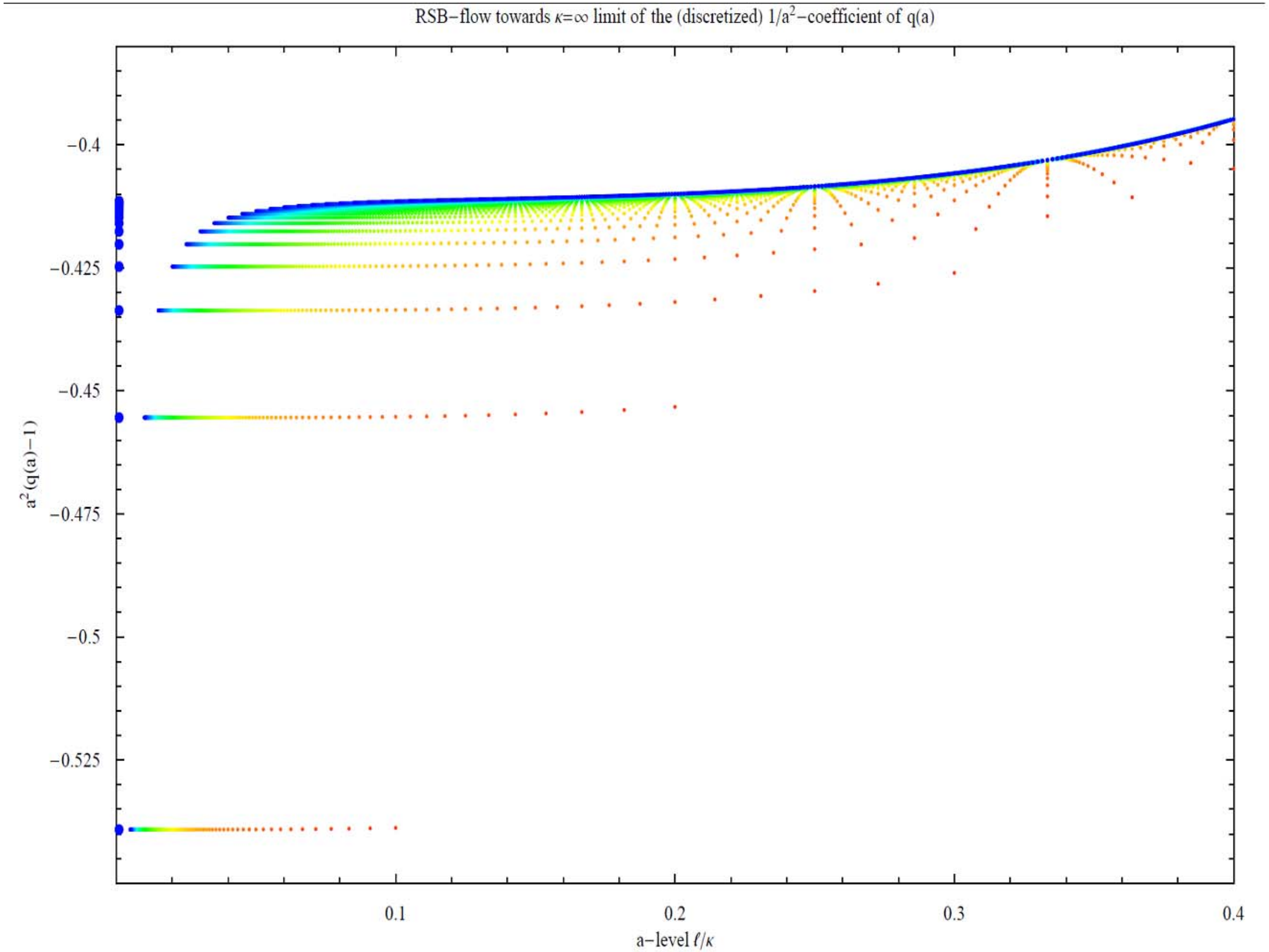
different divergent sub-classes of the variable ,a‘:

discrete Parisi-regime $x>0$ is not continuous at zero temperature and:

nonlinear effects at CP2 prevent discrete derivative to agree with continuous regime

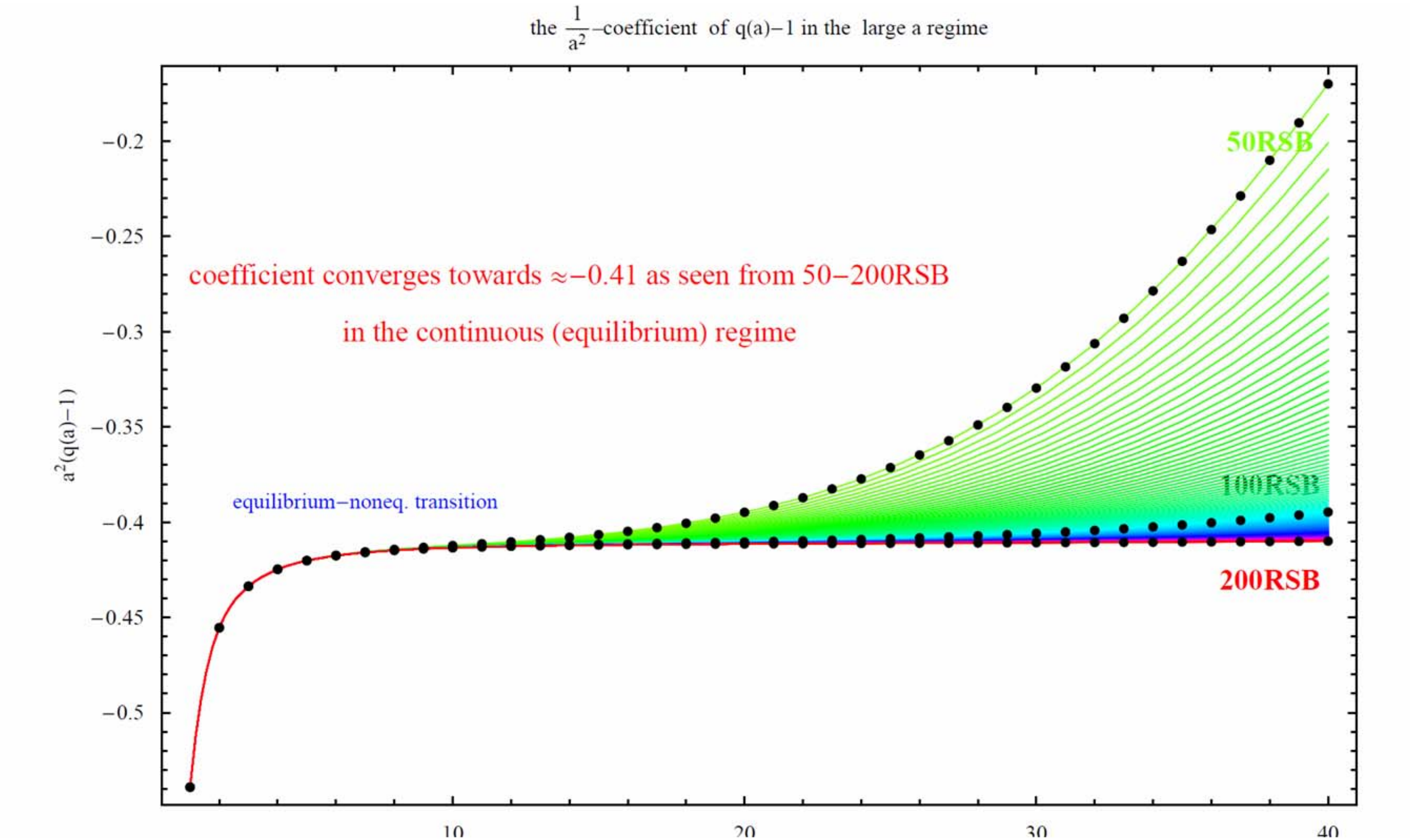
transition lies at infinite a and $x=0$

multivalued derivative of q(a) in the large-a limit

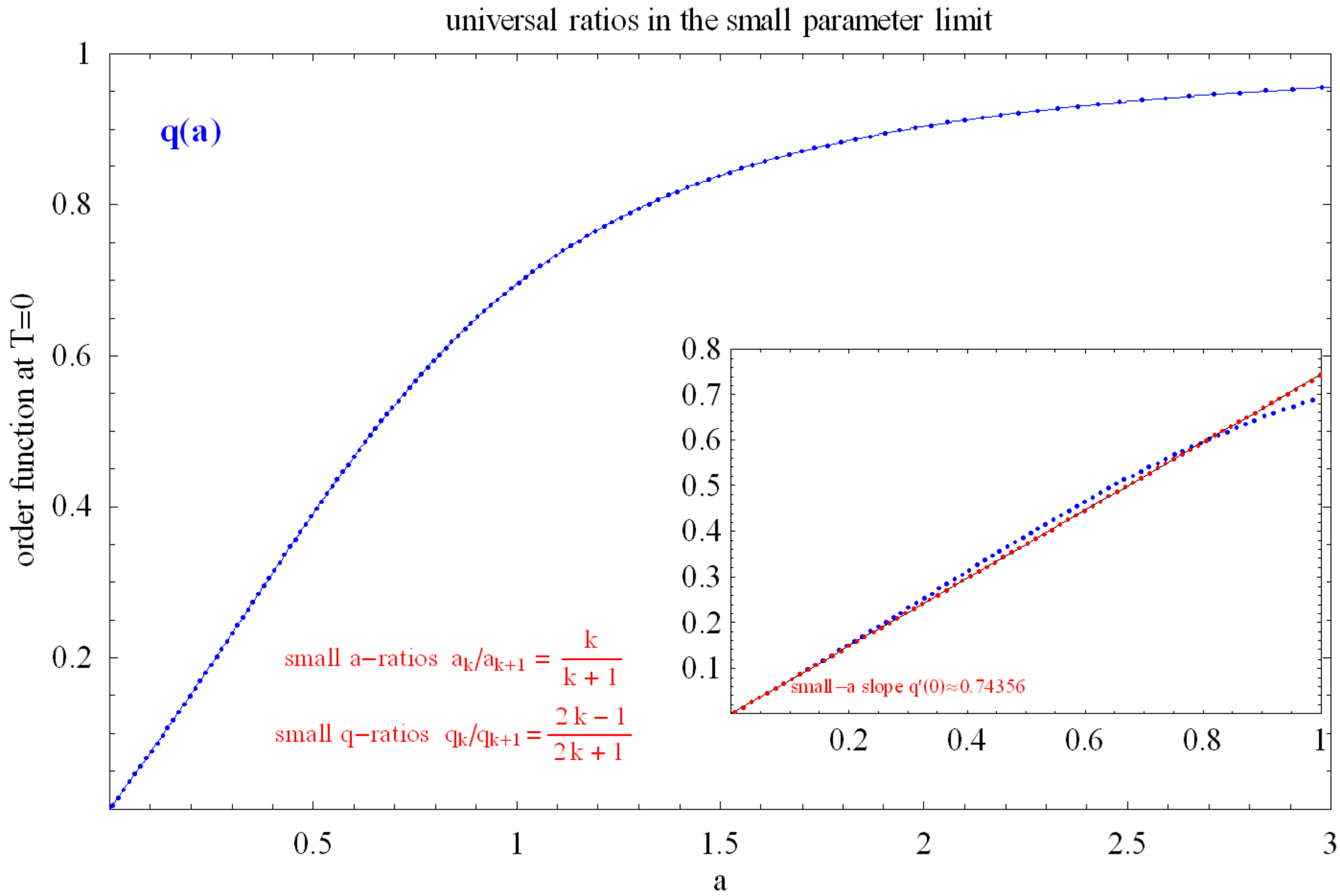


$$\begin{pmatrix} -0.539128 \\ -0.45542 \\ -0.433662 \\ -0.424745 \\ -0.420208 \\ -0.417582 \\ -0.415923 \\ -0.414808 \\ -0.414022 \\ -0.413447 \\ -0.413013 \\ -0.412679 \\ -0.412415 \\ -0.412203 \\ -0.41203 \\ -0.411887 \\ -0.411768 \\ -0.411668 \\ -0.411582 \\ -0.411509 \\ -0.411445 \\ -0.41139 \\ -0.411341 \\ -0.411299 \\ -0.411261 \\ -0.411227 \\ -0.411197 \\ -0.41117 \\ -0.411146 \\ -0.411124 \end{pmatrix}$$

transition from continuous large a regime into discontinuous Parisi regime



infinite-RSB-extrapolated slope from discrete spectra (q- and a-ratios) compared with 200-RSB order parameter function (confluent hypergeometric function model)



scaling theory

scaling near CP1

Parisi box sizes:

$$m_l(\kappa, T) \equiv a_l(\kappa, T)\, T = a_l(\kappa, 0)T + a_l'(\kappa, 0)T^2 + O(T^3)$$

$$a_l(\kappa, T) = \kappa^{5/3} f_{a_l}(T/\kappa^{-5/3})$$

$$f_{a_l}(x) = \frac{c_{0,l} + c_{1,l}x}{1 + d_{1,l}x + d_{2,l}x^2}$$

nonequilibrium susceptibility:

$$\chi_1(\kappa, T) = \kappa^{-5/3} f_1(T/\kappa^{-5/3})$$

entropy:

$$S_s(\kappa, T) = \kappa^{-10/3} f_S(T^2/\kappa^{-10/3})$$

scaling near CP2

Plateau height (and –width):

$$q_i(\kappa, H, T=0) = \frac{1}{\kappa} f_i\left(\frac{H^{2/3}}{1/\kappa}\right)$$

free energy contribution:

$$F_s^{(\mathcal{CP}2)}(\kappa, H) = \kappa^{-5} f_{cp2}(H^{2/3}/\kappa^{-1})$$

free energy F[q(a)] as a functional of q(a)

contains CP1, CP2 and regular contributions

$$F(\kappa, T, H) = F_{reg}(\kappa, H, T) + F_s^{(\mathcal{CP}1)}(\kappa, T) + F_s^{(\mathcal{CP}2)}(\kappa, H)$$

$$F_s^{(\mathcal{CP}1)}(\kappa, T) = \kappa^{-5} f_{cp1}(T/\kappa^{-5/3})$$

$$F_s^{(\mathcal{CP}2)}(\kappa, H) = \kappa^{-5} f_{cp2}(H^{2/3}/\kappa^{-1})$$

Conclusion:

RSB is a 1D-critical T=0 fluctuation theory by itself

new exact analytic approach

partial differential equation for the
exponential-correction function expC
becomes an unusual Burgers-type equation

however non-linearizable by the
Cole-Hopf-Fedyanin transformation

separating asymptotically simple from nontrivial parts of the free energy

$$\delta F_\kappa = \frac{T}{m_\kappa} \int_{\kappa+1}^{G} \log f_\kappa (h_{\kappa+1}) \quad \text{with} \quad f_{\alpha+1}(h_{\alpha+2}) = \int_{\alpha+1}^{GE} f_\alpha (h_{\alpha+1})$$

$$f_1(h_2) = \int_1^{GE} f_0(h_1) = \int_1^{G} \left(2\cosh\left(\frac{h_1}{T}\right)\right)^{a_1 T} = \int_1^{GE} \exp\left(\,|\,h_1\,| + \mathbb{e}\mathrm{xp}C_0(h_1)\right)$$

by iteration the f_α − recursion can be written as a $\mathbb{e}\mathrm{xp}C_\alpha$ − recursion

$$\mathbb{e}\mathrm{xp}C_\alpha(h_{\alpha+1}) = \frac{1}{a_\alpha} \log \int_{-\infty}^{\infty} \frac{d h_\alpha}{\sqrt{2\pi\delta q_\alpha}}\, \mathbb{e}^{-\frac{1}{2} a_\alpha^2 \delta q_\alpha + a_\alpha (|h_\alpha| - |h_{\alpha+1}|) - \frac{(h_{\alpha+1}-h_\alpha)^2}{2\delta q_\alpha} + a_\alpha \mathbb{e}\mathrm{xp}C_{\alpha-1}(h_\alpha)}$$

in the RSB limit a nonlinear PDE results for expC(a,h) within the continuum regime

PDE for continuous equilibrium regime

$$\partial_a \, \mathcal{e}\mathrm{xpC} = -\frac{\partial_a q(a)}{2}\left[\partial_h^2 \mathcal{e}\mathrm{xpC} + 2\, a\, \partial_h \mathcal{e}\mathrm{xpC} + (\partial_h \mathcal{e}\mathrm{xpC})^2\right]$$

initial conditions

$$\partial_h \exp\mathcal{C}(a,h)|_{h=0^+} = -1; \quad \exp\mathcal{C}(a,\infty) = 0$$
$$\exp\mathcal{C}(\beta,h) = T\log\left(1+\exp\left(-2\beta h\right)\right)$$

free energy at zero temperature in terms of expC

$$F_\kappa(T=0) = \frac{1}{4}\sum_{\alpha=1}^{\kappa} a_\alpha\left((q_\alpha - 1)^2 - (q_{\alpha+1} - 1)^2\right) \quad - \quad \int_h^G \left(\mathcal{e}\mathrm{xpC}_\kappa(h) + |\,h\,|\right)$$

$$\xrightarrow[\kappa\to\infty]{} \quad F_{exact}(T=0) = -\frac{1}{4}\int_0^\infty \left(1 - q(a)^2\right) da \; - \; \mathcal{e}\mathrm{xpC}(a=0, h=0)$$

$$F_\kappa = F_{\kappa=\infty} + \frac{\mathrm{const}}{(\kappa + \mathrm{const})^4} + \ldots, \qquad F_{\kappa\to\infty} = -0.763166726566547\ldots$$

PDE for the nontrivial correction function expC(a,h):

$$\partial_a \, e\mathrm{xp}C =$$

$$-\frac{1}{2}\, q'(a) \left\{ \partial_h^2 e\mathrm{xp}C + 2\, a\, \partial_h e\mathrm{xp}C + a \left(\partial_h e\mathrm{xp}C\right)^2 \right\}$$

transformed into diffusion equation with nonlinear gradient term

$$\frac{2}{q'(a)}\, \partial_a Y_1 + \partial_h^2 Y_1 + a \left(\partial_h Y_1\right)^2 = a$$

$$\longrightarrow \quad \partial_q Y_2 = \frac{1}{2}\, \partial_h^2 Y_2 + \frac{1}{2}\, a(q) \left(\partial_h Y_2\right)^2$$

where $Y_2(a, h) = Y_1(a, h) - \chi(a)$ and $\chi(a) = \int da\, a\, q'(a)$

KPZ – equation for interface growth in 1 + 1 dimension

$$\partial_t \phi = \nu\, \partial_x^2 \phi + \frac{\lambda}{2} (\partial_x \phi)^2 + \eta(x, t) \quad \text{with } \delta\text{ – correlated noise :}$$

$$\langle \eta(x, t)\, \eta(x', t') \rangle = \gamma\, \delta(x - x')\, \delta(t - t')$$

derived from a random – noise driven Burgers equation

$$\partial_t u = \nu\, \partial_x^2 u + u\, \partial_x u$$

which by means of $u = \partial_x \psi$ is rewritten as

$$\partial_t \partial_x \psi = \partial_x \left(\frac{1}{2} (\partial_x \psi)^2 + \nu\, \partial_x^2 \psi \right)$$

and linearized by a Cole Hopf (– Fedyanin) transformation

$$\psi = 2\, \nu \log \phi$$

becoming a diffusion equation for ϕ

analogy and difference
between KPZ-eq and spin glass PDE

as KPZ-analogy the spin-glass PDE
contains a pseudotime-dependent nonlinear coupling
a time-dependent Cole-Hopf transformation
removes nonlinear gradient term
in favour of a nonlinear logarithmic potential term
hence: PDE remains nonlinear

scaling properties essentially different, since spin glass PDE is an equilibrium equation of the RSB-limit and can be written as a noise-free pseudo-dynamical Halperin-Hohenberg-equation

conclusions

the RSB-limit (so-called ‚continuum limit‘) is not fully continuous at T=0

a nonlinear PDE exists however even at T=0

turns into 1D KPZ-like equation by adding a noise term
for nonequilibrium transition between different orders of RSB

critical magnetic field-behaviour resembles EW linear KPZ fixed point behaviour, thermal behaviour corresponds to KPZ nontrivial fixed point (yet differs in detail)

spin glass shows crossover from 1D-RSB double-criticality to mean-field SK behaviour

scaling exponents shows 1D-character, resembles directed polymers, ASEP…

scale invariances as relevant symmetries of the low-T SK spin glass phase

T=0 RSB theory comparable with droplet theory as a T=0 critical phenomenon