Dyson's non-intersecting

Brownian motions

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1. GUE and Dyson's non-intersecting Brownian motions

(Wigner, Dyson & Mehta , 50's and 60's)

(Karlin-McGregor '59)

GUE: Hermitian matrices $M \in \mathcal{H}_n$, with independent, identically distributed Gaussian entries

$$P(M \in dM) = \frac{1}{Z_n} e^{-\operatorname{Tr} M^2} dM, \quad M \in \mathcal{H}_n,$$

If $M \in \mathcal{H}_n$ is chosen at random

$$\mathbb{P}(\text{all eigenvalues } \in [x_1, x_2]) = \frac{1}{Z_n} \int_{[x_1,x_2]^n} \Delta(\lambda_1, \ldots, \lambda_n)^2 \prod_1^n e^{-\lambda_i^2} d\lambda_i$$

Dyson ('62): Put dynamics in the system with transition density from M_0 to M_1:

$$P(t, M_0, M_1) = \frac{1}{Z_n(1-c^2)^{n^2/2}} e^{-\frac{\mathrm{Tr}(M_1 - cM_0)^2}{a^2(1-c^2)}} \quad \text{with} \quad c := e^{-t/a^2}.$$

$$\Updownarrow$$

Dyson ('62): Put dynamics in the system with transition density from M_0 to M_1:

$$P(t, M_0, M_1) = Z^{-1} \frac{1}{(1-c^2)^{n^2/2}} e^{-\frac{\mathrm{Tr}(M_1 - cM_0)^2}{a^2(1-c^2)}} \quad \text{with} \quad c := e^{-t/a^2}.$$

$$\Updownarrow$$

n^2 independent **Ornstein-Uhlenbeck diffusions** on the n^2 free parameters M_ν of the Hermitian matrix M:

$$\frac{\partial P}{\partial t} = \sum_{\nu=1}^{n^2} \left(\frac{1}{4}(1+\delta_\nu) \frac{\partial^2}{\partial M_\nu^2} + \frac{1}{a^2} \frac{\partial}{\partial M_\nu} M_\nu \right) P$$

$$\Updownarrow$$

n^2 independent **Ornstein-Uhlenbeck diffusions** on the n^2 entries M_ν:

$$\frac{\partial P}{\partial t} = \sum_{\nu=1}^{n^2} \left(\frac{1}{4}(1+\delta_\nu) \frac{\partial^2}{\partial M_\nu^2} + \frac{1}{a^2} \frac{\partial}{\partial M_\nu} M_\nu \right) P$$

$$\Updownarrow$$

The eigenvalues $\lambda_1(t) < \ldots < \lambda_n(t)$ of M evolve according to a **diffusion**

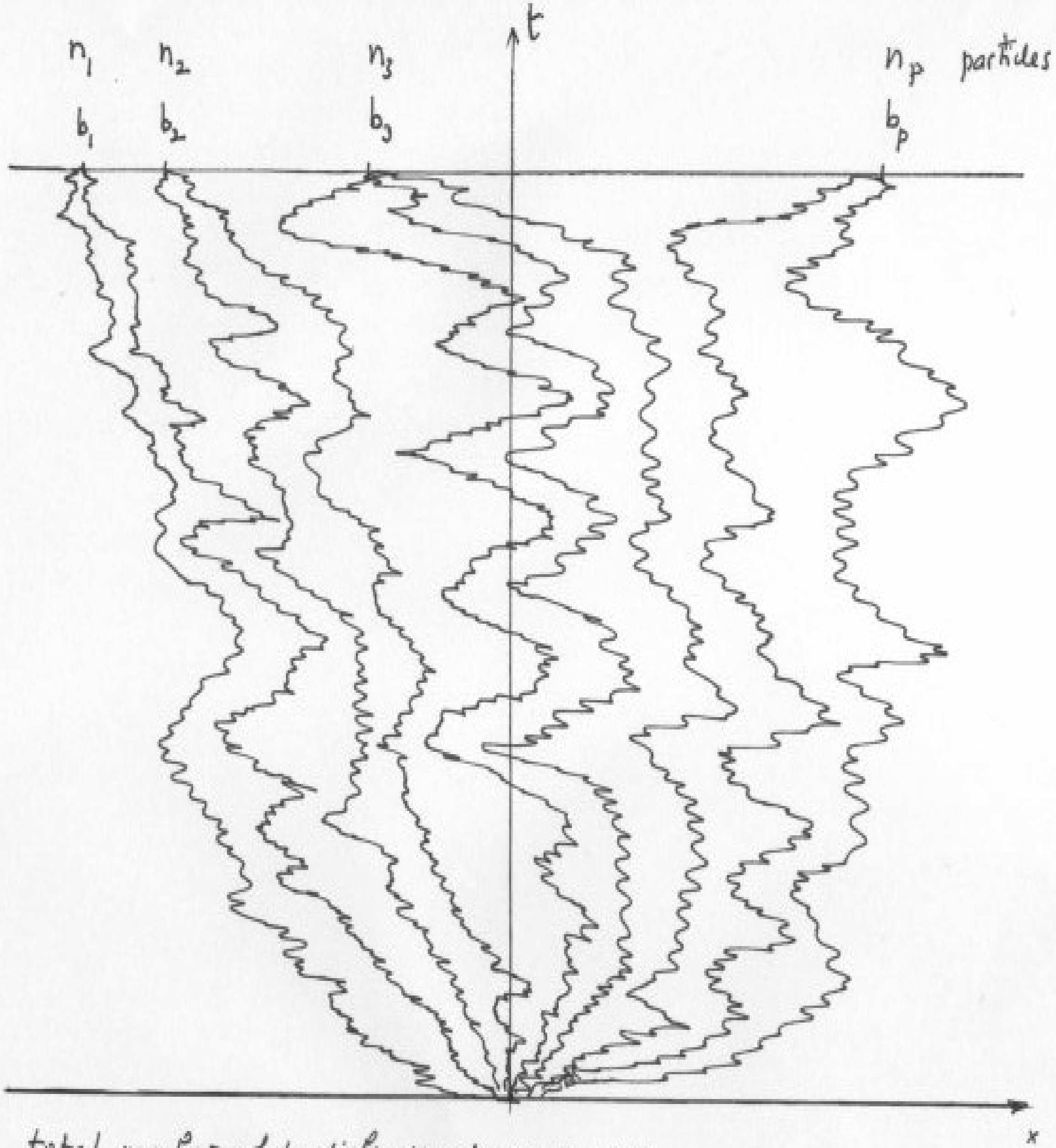
- with transition density $p(t, \mu, \lambda)$, satisfying:

$$\frac{\partial p}{\partial t} = \frac{1}{2} \sum_1^n \frac{\partial}{\partial \lambda_i} \Phi(\lambda) \frac{\partial}{\partial \lambda_i} \frac{1}{\Phi(\lambda)} p,$$

- equilibrium measure $\Phi(\lambda) = \frac{1}{Z} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2 \prod_1^n e^{-\frac{\lambda_i^2}{a^2}}$,

1. Non-intersecting Brownian motions, leaving from a point and forced to one or several points

Karlin-McGregor '59, Dyson '62, Ueno-Takasaki '84, P. Zinn-Justin '97, '98, Eynard-Mehta '98, Grabiner '99, Johansson '01, Aptekarev-Bleher-Kuijlaars '04, Tracy-Widom '03 and '05, Bleher-Kuijlaars '04, Adler-PvM '05, Kuijlaars-Daems-Veys '05, Adler-PvM-Vanhaecke '06



total number of particles $n = n_1 + n_2 + \cdots + n_p$

- n non-intersecting Brownian motions $X_1(t), \ldots, X_n(t)$ $\quad (\sum_1^p n_i = n)$

forced to leave from 0

forced to arrive at $b := (\overbrace{b_1, b_1, \ldots, b_1}^{n_1}, \overbrace{b_2, b_2, \ldots, b_2}^{n_2}, \ldots, \overbrace{b_p, b_p, \ldots, b_p}^{n_p})$

The probability

$$\mathbb{P}_n^{(b_1,\ldots,b_p)} \left(\textbf{all } (x_1(t) < \ldots < x_n(t)) \in E \;\middle|\; \begin{array}{c} \textbf{all } x_j(0) = 0 \\ n_1 \textbf{ paths end up at } b_1 \textbf{ at } t = 1 \\ \vdots \\ n_p \textbf{ paths end up at } b_p \textbf{ at } t = 1 \end{array} \right)$$

has 4 very different descriptions !

Brownian motions evolves locally like

$$p(t; x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

- Determinant of a block moment matrix
- Gaussian model with external potential
- Fredholm determinant
- PDE for the transition probability: *A near-Wronskian !*

Consider now the probability, as a function of

- the boundary points of E
- the target points b_i, with a linear dependence $\sum_1^p c_i b_i = 0$, with $\sum_1^p c_i = 1$

$$\log \mathbb{P}_n^{(b_1,\dots,b_p)} \left(\text{all } (x_1(t) < \ldots < x_n(t)) \in E \;\middle|\; \begin{array}{c} \text{all } x_j(0) = 0 \\ n_1 \text{ paths end up at } b_1 \text{ at } t = 1 \\ \vdots \\ n_p \text{ paths end up at } b_p \text{ at } t = 1 \end{array} \right)$$

satisfies a non-linear PDE in the boundary points of the interval E and in the target points b_i(*nearly a Wronskian!!*)

$$\det \begin{pmatrix} F_1 & F_2 & F_3 & \ldots & F_p & 0 \\ \partial F_1 & \partial F_2 & \partial F_3 & \ldots & \partial F_p & G_1 \\ \partial^2 F_1 & \partial^2 F_2 & \partial^2 F_3 & \ldots & \partial^2 F_p & G_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \partial^p F_1 & \partial^p F_2 & \partial^p F_3 & \ldots & \partial^p F_p & G_p \end{pmatrix} = 0, \qquad \partial := \left\{ \begin{array}{l} \textit{sum of} \\ \textit{partials} \\ \textit{in the} \\ \textit{boundary} \\ \textit{points of } E \end{array} \right\}$$

(Adler-Vanhaecke-PvM CMP '08)

where

$$F_\ell \;:=\; \left(\partial_\ell^{(b)} + c_\ell \partial\right) \partial \log \mathbb{P}_n + n_\ell$$

$$G_{\ell+1} \;:=\; \partial G_\ell \;\; + \sum_{j=1}^{p} (\partial^\ell F_j) \left(\partial \frac{H_j^{(1)}}{F_j} - \partial_j^{(b)} \frac{H_j^{(2)}}{F_j} \right), \qquad G_0 = 0$$

with (remember $\sum_1^p c_i b_i = 0$, so that $\sum_{\ell=1}^p \nabla_\ell^{(b)} = 0$)

$$\partial_\ell^{(b)} \;:=\; c_\ell \sum_1^{p-1} \frac{\partial}{\partial b_i} - \frac{\partial}{\partial b_\ell}(1 - \delta_{\ell p})$$

What happens to this probability, when $\#\{\text{particles}\} = n \to \infty$?

- Airy process
- Pearcey process
- Airy process, with outliers
- Pearcey process with inliers,. . .

Define now the **matrix kernels** for arbitrary t_1 and t_2,

$$\mathbb{K}(t_1,x;t_2,y) := \begin{pmatrix} K(t_1,x;t_1,y) & K(t_1,x;t_2,y) \\ \\ K(t_2,x;t_1,y) & K(t_2,x;t_2,y) \end{pmatrix}$$

Examples will be discussed, leading to a process $\mathcal{P}(t)$:

$$\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{P}(t_k)\cap E_k=\emptyset\}\right) = \det\Big(I-(\chi_{E_i}K(t_i,.;t_j,.)\chi_{E_j})_{1\le i,j\le m}\Big)$$

(i) Airy kernel $= K^{\mathcal{A}}(t_1, x; t_2, y)$

$$= \frac{1}{(2\pi i)^2}\iint \frac{dV\,dU}{(U+t_2)-(V+t_1)}\,\frac{e^{-\frac{U^3}{3}+yU}}{e^{-\frac{V^3}{3}+xV}}\prod_{k=1}^{m}\left(\frac{V-a_k+t_1}{U-a_k+t_2}\right)\left(\frac{U-b_k+t_2}{V-b_k+t_1}\right)$$

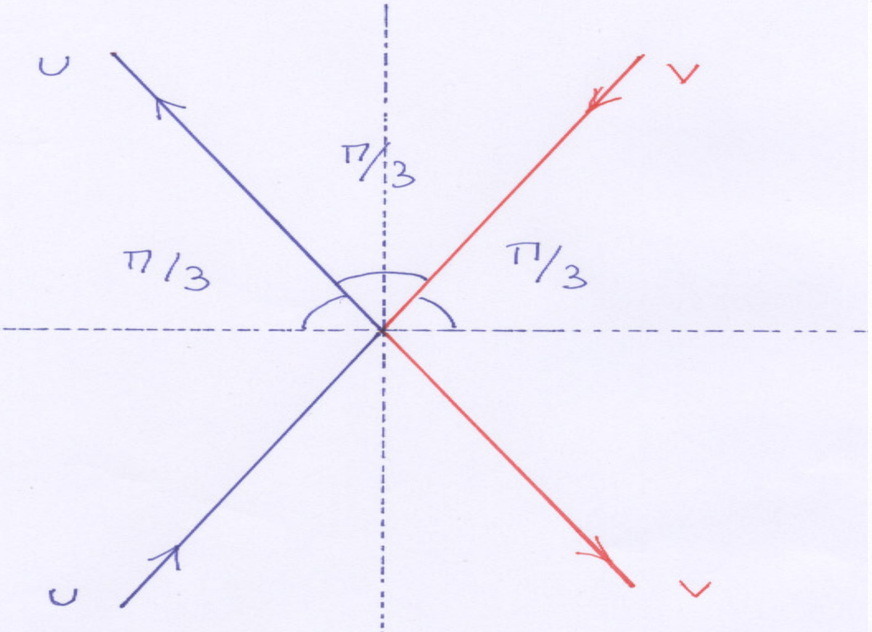
$$-\frac{\mathbb{I}(t_1<t_2)}{\sqrt{4\pi(t_2-t_1)}}e^{\frac{1}{12}(t_2-t_1)^3-\frac{(x-y)^2}{4(t_2-t_1)}-\frac{1}{2}(t_2-t_1)(y+x)}$$

(ii) Pearcey kernel $= K^{\mathcal{P}}(t_1, x; t_2, y)$

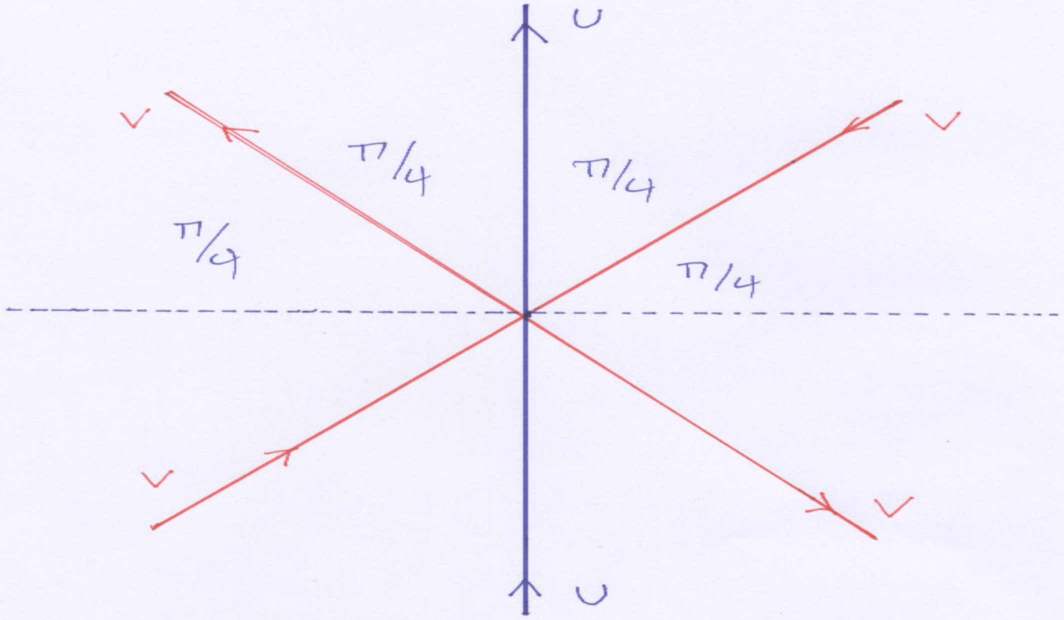
$$= -\frac{1}{4\pi^2}\iint \frac{dV\,dU}{U-V}\;\frac{e^{-\frac{U^4}{4}+\frac{t_2U^2}{2}-Uy}}{e^{-\frac{V^4}{4}+\frac{t_1V^2}{2}-Vx}}\prod_{k=1}^{m}\left(\frac{U-b_k+t_2}{V-b_k+t_1}\right)-\frac{\mathbb{I}(t_1<t_2)}{\sqrt{2\pi(t_2-t_1)}}e^{-\frac{(x-y)^2}{2(t_2-t_1)}}$$

(iii) Quintic kernel $= K^{\mathcal{Q}}(\theta; x, y)$

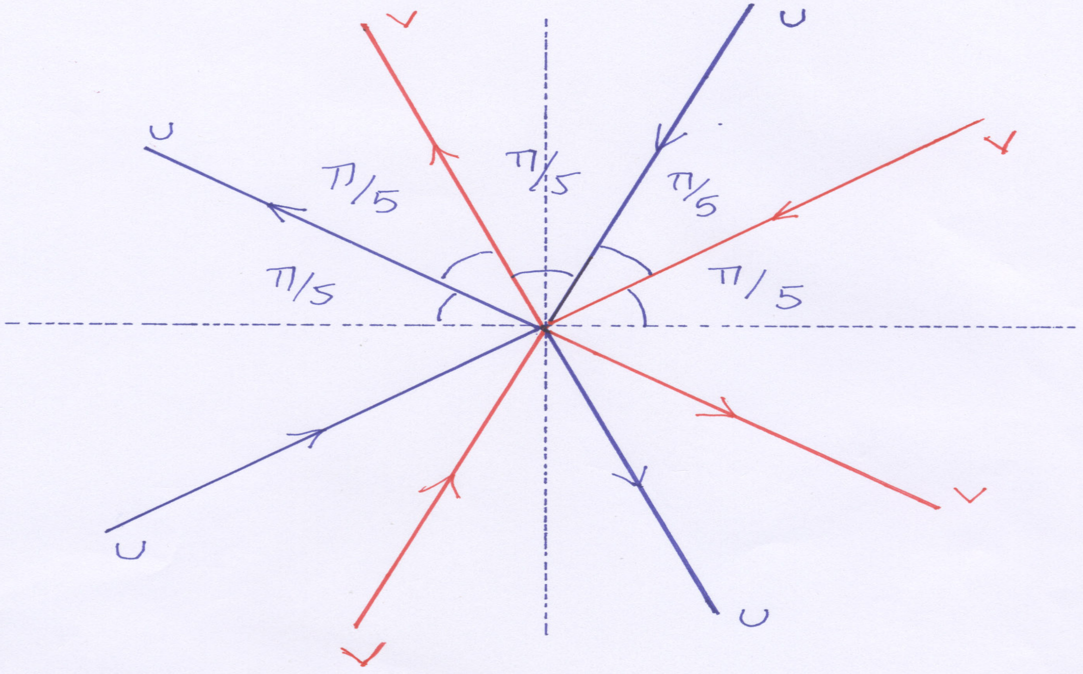
$$= \frac{1}{(2\pi i)^2}\iint \frac{dV\,dU}{V-U}\;\frac{e^{-\frac{2}{5}U^5+\frac{1}{3}\theta U^3-Ux}}{e^{-\frac{2}{5}V^5+\frac{1}{3}\theta V^3-Vy}},$$



Airy contours



Pearcey contours



Quintic contours

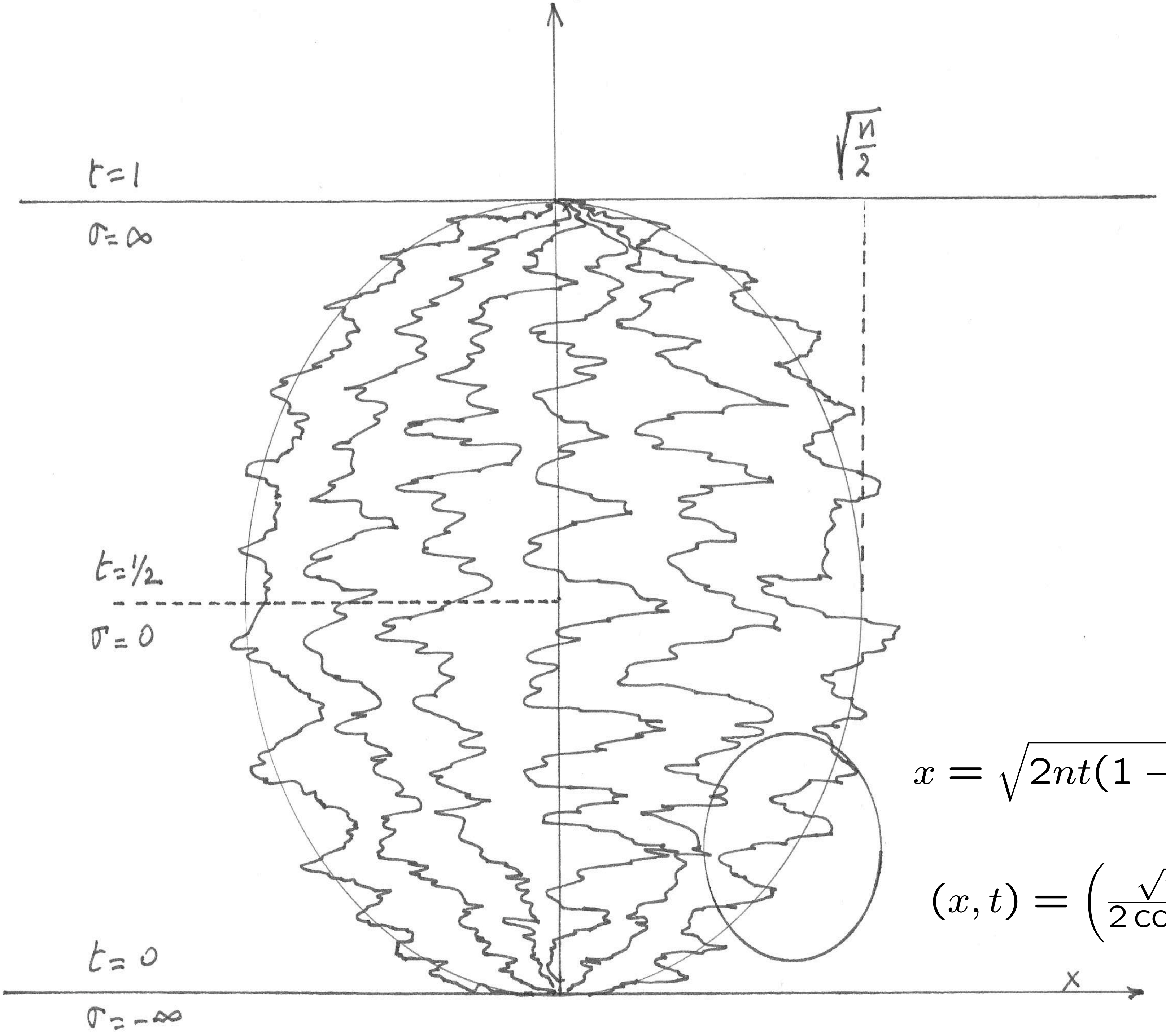
2. The Airy process $A(\tau)$

Non-intersecting Brownian motions leaving from 0 and returning to 0 at $t = 1$:

$$\mathbb{P}\left(\text{all } x_j(t) \in E^c \text{ for } 1 \le j \le n \;\middle|\; \begin{array}{l} \text{all } x_j(0) = 0 \\ \\ \text{all } x_j(1) = 0 \end{array} \right).$$

Let $n \to \infty$:

(Prähofer-Spohn ’02, Johansson ’03, ’05, Tracy-Widom ’04, Adler-PvM ’05),



$$x = \sqrt{2nt(1-t)}$$

$$(x,t) = \left(\frac{\sqrt{2n}}{2\cosh\sigma}, \quad \frac{e^{\sigma}}{2\cosh\sigma}\right)$$

Look through a microscope at **the fluctuations of the paths** about any point on the curve, for very large n:

$$(x,t)=\left(\frac{\sqrt{2n}}{2\cosh\sigma},\quad\frac{e^{\sigma}}{2\cosh\sigma}\right)\in\text{curve:}\qquad x=\sqrt{2nt(1-t)}$$

New process $\mathcal{A}(\tau)$ in the neighborhood of this point $(x,t)\in$ curve :
Pick times $0<\tau_1<\ldots<\tau_m<1$ and intervals $E_1,\ldots,E_m$.

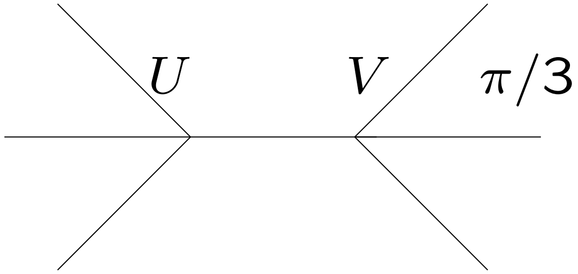
$$\lim_{n\to\infty}\mathbb{P}^0_{Br}\left(\bigcap_{k=1}^{m}\left\{\text{all } x_i\left(\frac{e^{\sigma+\tau_k n^{-1/3}}}{2\cosh(\sigma+\tau_k n^{-1/3})}\right)\in\frac{\sqrt{2n}+\frac{E_k^c}{\sqrt{2}n^{1/6}}}{2\cosh(\sigma+\tau_k n^{-1/3})}\right\}\right)$$

$$=:\det\left(I-(\chi_{E_i}K^{\mathcal{A}}_{\tau_i\tau_j}\chi_{E_j})_{1\leq i,j\leq m}\right),\qquad\text{independent of }\sigma!!!$$

$$=\mathbb{P}\left(\bigcap_{k=1}^{m}\{\mathcal{A}(\tau_k)\cap E_k=\emptyset\}\right)$$

with kernel:

$$K^{\mathcal{A}}(\tau_i,\xi_i;\tau_j,\xi_j) = -\frac{\mathbb{I}(\tau_j>\tau_i)}{\sqrt{4\pi(\tau_j-\tau_i)}}e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_j-\tau_i)}-\frac{1}{2}(\tau_j-\tau_i)(\xi_j+\xi_i)+\frac{1}{12}(\tau_j-\tau_i)^3}$$
$$+\frac{1}{(2\pi i)^2}\int_{\Gamma}dU\int_{\Gamma'}dV\,\frac{e^{-U^3/3+\xi_jU}}{e^{-V^3/3+\xi_iV}}\quad\frac{1}{(U+\tau_j)-(V+\tau_i)}.$$



- $\mathbb{P}(\sup \mathcal{A}(\tau) \le x) = e^{-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha} =: \mathcal{F}(x)$ (Tracy-Widom distribution '9

(time-independent: stationary process)

where $g(\alpha)$ is the Hastings-MacLeod solution of Painlevé II:

$$g'' = \alpha g + 2g^3, \qquad \text{with} \quad g(\alpha) \cong A(\alpha) = \int_{-\infty}^{\infty} e^{\frac{1}{3} i u^3 + i \alpha u} \frac{du}{2\pi} \quad \text{for} \quad \alpha \nearrow \infty.$$

(Airy function)

- $\mathbb{Q}(s; x, y) := \log \mathbb{P}\left(\sup \mathcal{A}(\tau_1) \le x + y, \ \sup \mathcal{A}(\tau_2) \le x - y\right), \quad s = \dfrac{\tau_2 - \tau_1}{2} > 0$

satisfies the PDE ($\{\ ,\ \}$ is a Wronskian)

$$2s \frac{\partial^3 \mathbb{Q}}{\partial s \partial x \partial y} = \left(2s^2 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \left(\frac{\partial^2 \mathbb{Q}}{\partial y^2} - \frac{\partial^2 \mathbb{Q}}{\partial x^2}\right) + \left\{\frac{\partial^2 \mathbb{Q}}{\partial x \partial y}, \frac{\partial^2 \mathbb{Q}}{\partial x^2}\right\}_x .$$

(Adler-PvM, '05),

with "initial" condition: $\lim_{s \to \infty} \mathbb{Q}(s; x, y) = \log \mathcal{F}(x + y) + \log \mathcal{F}(x - y)$.

3. The Pearcey process $\mathcal{P}(\tau)$

$-\infty < b < a < \infty$

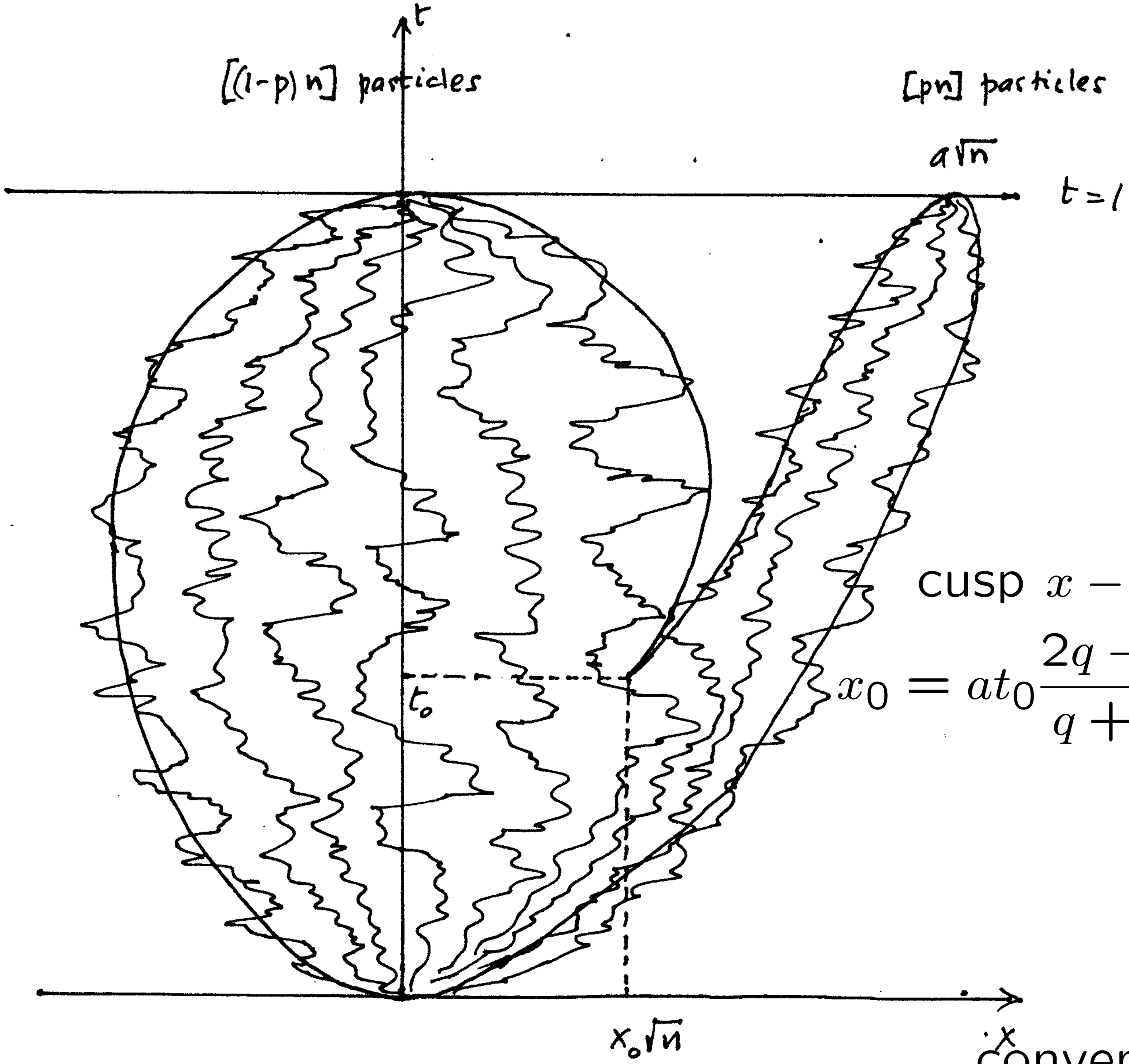
$$\mathbb{P}^{ba}_{Br}\left(\text{all } x_j(t) \in E^c\right)$$

$$:= \mathbb{P}\left(\text{all } x_j(t) \in E^c \;\middle|\; \begin{array}{c} \text{all } x_j(0) = 0 \\ [(1-p)n] \text{ paths end up at } b\sqrt{n} \text{ at } t=1 \\ [pn] \text{ right paths end up at } a\sqrt{n} \text{ at } t=1 \end{array}\right).$$

Let $n \to \infty$:

Pastur '72, Brézin-Hikami '96-98, P. Zinn-Justin '97-98, Johansson '01, Bleher-Kuijlaars '04, Tracy-Widom '05, Okounkov-Reshetikhin '05, Adler-PvM '05, Adler-Orantin-PvM '08

Set $b = 0$:



cusp $x - x_0 = 2\left(\frac{t-t_0}{3}\right)^{3/2}$ at

$$x_0 = at_0\frac{2q-1}{q+1}, \ t_0 = \frac{1}{1+2\left(\frac{a(q^3-1)}{q+1}\right)^2}.$$

convenient parametrization:

$$\frac{1-p}{p} =: q^3$$

Look through a microscope at **the fluctuations of the paths** about the cusp, for very large n: (take $b=0$)

$$\textbf{cusp}: \qquad x - x_0 = 2\left(\frac{t-t_0}{3}\right)^{3/2}$$

New process $\mathcal{P}(\tau)$ in the neighborhood of the cusp : (universality!)

$$\lim_{n\to\infty} \mathbb{P}_n^{(0,a\sqrt{n})}\left(\text{all } x_j\left(t_0 + (c_0\mu)^2\frac{2\tau}{n^{1/2}}\right) \in x_0 n^{1/2} + c_0 A\tau + c_0\mu\frac{E^c}{n^{1/4}}\right)$$

$$= \det\left(I - K_\tau^{\mathcal{P}}\right)_E \qquad \text{(independent of } pn \text{ and } a\text{),}$$

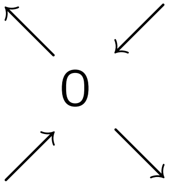
$$= \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(\tau)\cap E = \emptyset\right) \qquad (\textit{Pearcey process})$$

$$\mu^4 = \frac{q^2-q+1}{q}, \qquad c_0 := \frac{at_0\sqrt{q^2-q+1}}{q+1}, \qquad A = \sqrt{q} - \frac{(2q-1)t_0}{\sqrt{q}}$$

with Pearcey kernel: (for $\tau = \tau_1 = \tau_2$)

$$K^{\mathcal{P}}(\tau_1,\xi;\tau_2,\eta) := -\frac{1}{4\pi^2}\int_X dV \int_{-i\infty}^{i\infty} dU \quad \frac{e^{-\frac{U^4}{4}+\frac{\tau_2 U^2}{2}-U\eta}}{e^{-\frac{V^4}{4}+\frac{\tau_1 V^2}{2}-V\xi}}\, \frac{1}{U-V}$$

contour $X =$



Then $\mathbb{Q}(\tau;E) = \log \mathbb{P}^{\mathcal{P}}\left(\mathcal{P}(\tau)\cap E = \emptyset\right)$ satisfies (Adler-PvM '06, Adler-Orantin-PvM '08)

$$\frac{\partial^3 \mathbb{Q}}{\partial t^3} + \frac{1}{8}\left(\mathcal{E} - 2t\frac{\partial}{\partial t} - 2\right)\partial^2 \mathbb{Q} - \frac{1}{2}\left\{\partial^2 \mathbb{Q}, \partial\frac{\partial \mathbb{Q}}{\partial t}\right\}_{\partial} = 0.$$

with an *"initial condition" at $t = \infty$*, given by the Airy process.(see later)

For $E = [x,y]$, $\quad \partial = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, $\quad \mathcal{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

Moving out of the cusp should give a limit of Pearcey to Airy:

For $T_1, T_2 \to \infty$, such that

$$\frac{T_2 - T_1}{2(t_2 - t_1)} = (3T_1)^{1/3} + \frac{t_2 - t_1}{(3T_1)^{1/3}} + \frac{2t_1 t_2}{3T_1} + O\Big(\frac{1}{T^{5/3}}\Big);$$

the following approximation of the Airy process by the Pearcey process holds:

$$\mathbb{P}\left(\bigcap_{i=1}^{2}\left\{\frac{\mathcal{P}(T_i) - \frac{2}{27}(3T_i)^{3/2}}{(3T_i)^{1/6}} \cap (-E_i) = \emptyset\right\}\right)$$
$$= \mathbb{P}\left(\bigcap_{i=1}^{2}\{\mathcal{A}(t_i) \cap (-E_i) = \emptyset\}\right)\left(1 + O\Big(\frac{1}{T_1^{4/3}}\Big)\right), \text{ for } T_1 \to \infty.$$

(Much better approximation than at first sight!!)

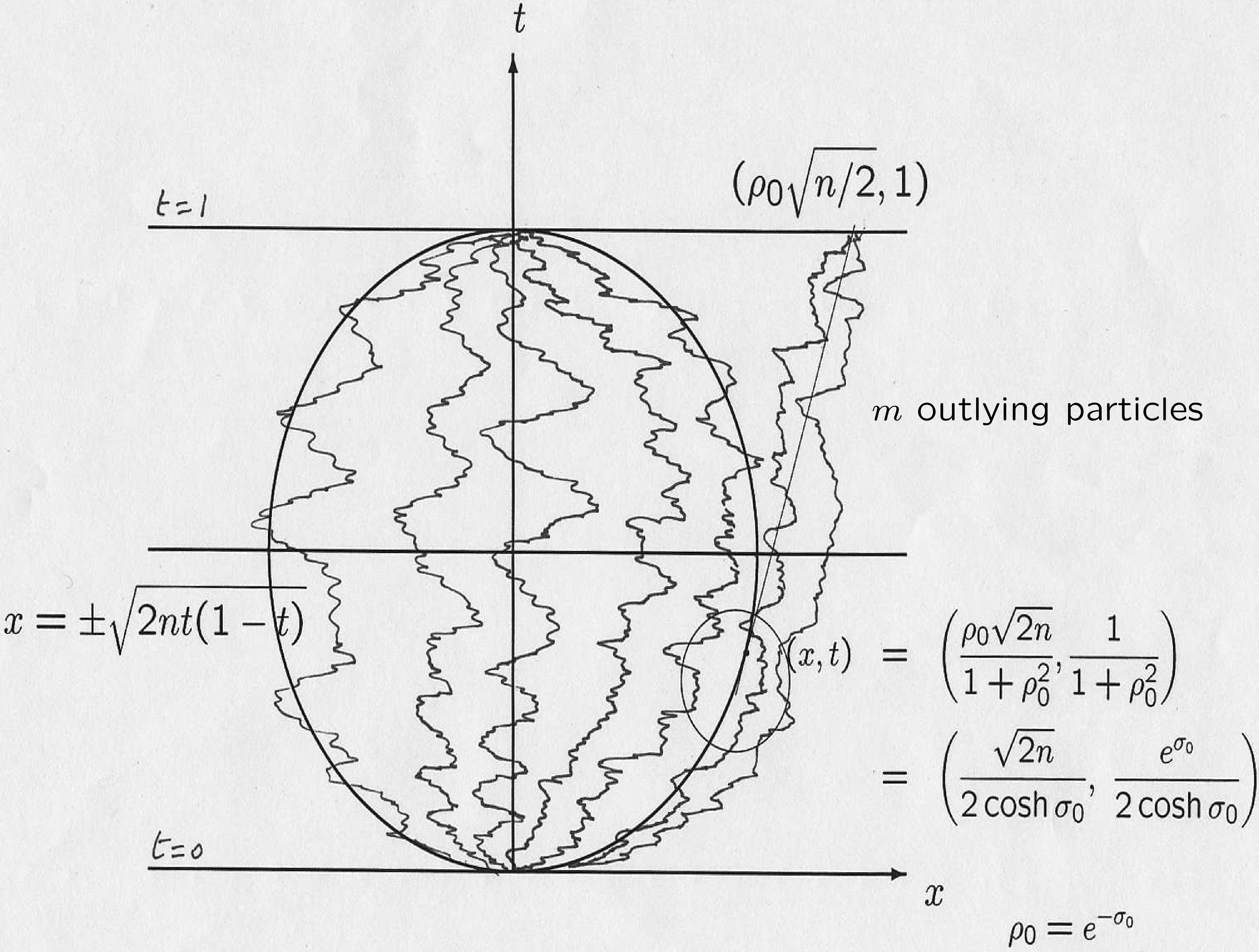
Adler-PvM-Cafasso '09

3. The Airy process $\mathcal{A}^{(m)}(t)$ with m outliers, leaving from 0

$$\mathbb{P}\left(\text{ all } x_j(t) \in E^c \,\middle|\, \begin{array}{c} \text{all } x_j(0) = 0 \\ m \text{ right paths end up at } a \text{ at } t = 1 \\ (n-m) \text{ paths end up at } 0 \text{ at } t = 1 \end{array} \right).$$

Keep m fixed and let $n \to \infty$:

(Adler, Delépine, PvM: CPAM '09.)


t
$t=1$
$(\rho_0\sqrt{n/2}, 1)$
m outlying particles
$x = \pm\sqrt{2nt(1-t)}$
(x,t)
$= \left(\frac{\rho_0\sqrt{2n}}{1+\rho_0^2}, \frac{1}{1+\rho_0^2}\right)$
$= \left(\frac{\sqrt{2n}}{2\cosh\sigma_0}, \frac{e^{\sigma_0}}{2\cosh\sigma_0}\right)$
$t=0$
x
$\rho_0 = e^{-\sigma_0}$

Look through the "Airy" microscope at **the fluctuations of the paths** about the point on the curve, for very large n:

$$(x,t)=\left(\frac{\sqrt{2n}}{2\cosh\sigma},\ \frac{e^{\sigma}}{2\cosh\sigma}\right)\in\ \text{curve}\qquad\text{for fixed}\ \left\{\begin{array}{l}\sigma\leq\sigma_0\\ \text{equivalently}\\ t\leq\frac{1}{1+\rho_0^2}\end{array}\right\}$$

$\Longrightarrow$ **Phase transition to new process** $\mathcal{A}^{(m)}(\tau)$: (Airy process with m outliers)

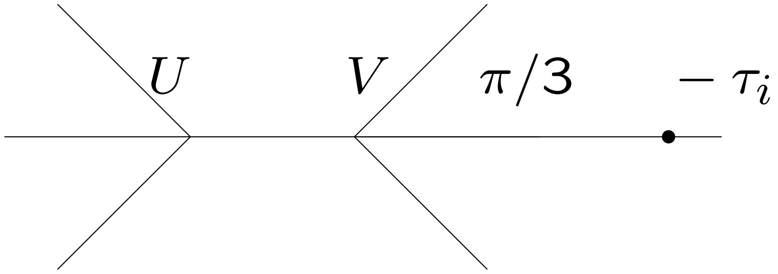
$$\lim_{n\to\infty}\mathbb{P}_{Br}^{(0,\rho_0\sqrt{n/2})}\left(\text{all }x_i\left(\frac{e^{\sigma+\tau n^{-1/3}}}{2\cosh(\sigma+\tau n^{-1/3})}\right)\in\frac{\sqrt{2n}+\frac{E^c}{\sqrt{2}n^{1/6}}}{2\cosh(\sigma+\tau n^{-1/3})}\right)$$

$$=\left\{\begin{array}{ll}\mathbb{P}(\mathcal{A}(\tau)\cap E=\emptyset)=\det\left(I-K^{\mathcal{A}}\right)_E & \text{for}\quad 0\leq\sigma<\sigma_0\\ \qquad\qquad(\textit{Airy process}) & \\ \mathbb{P}(\mathcal{A}^{(m)}(\tau)\cap E=\emptyset)=\det\left(I-K_m^{\mathcal{A}}\right)_E & \text{for}\quad\sigma=\sigma_0=\left\{\begin{array}{l}\text{point of}\\ \text{tangency}\end{array}\right.\end{array}\right.$$

(*This kernel appears in a different context in Baik, Ben Arous, Péché*) with (matrix Fredholm determinant)

$$\begin{aligned} K_m^{\mathcal{A}}(\tau_i,\xi_i;\tau_j,\xi_j) &= -\frac{\mathbb{I}(\tau_j>\tau_i)}{\sqrt{4\pi(\tau_j-\tau_i)}} e^{-\frac{(\xi_i-\xi_j)^2}{4(\tau_j-\tau_i)}-\frac{1}{2}(\tau_j-\tau_i)(\xi_j+\xi_i)+\frac{1}{12}(\tau_j-\tau_i)^3} \\ &\quad + \frac{1}{(2\pi i)^2}\int_{\Gamma} dU \int_{\Gamma'} dV\, \frac{e^{-U^3/3+\xi_j U}}{e^{-V^3/3+\xi_i V}} \quad \frac{\left(\frac{U+\tau_j}{V+\tau_i}\right)^m}{(U+\tau_j)-(V+\tau_i)}. \end{aligned}$$

$$K^{\mathcal{A}}(\tau_i,\xi_i;\tau_j,\xi_j) = K_m^{\mathcal{A}}(\tau_i,\xi_i;\tau_j,\xi_j)\Big|_{m=0}$$



Then

$$\mathbb{Q}(\tau,x) := \log \mathbb{P}(\sup \mathcal{A}^{(m)}(\tau) \le x) = \log\det\left(I - K_\tau^{(m)}\right)_{(x,\infty)}$$

satisfies a PDE ($\{f,g\}_x$ is the Wronskian with regard to x.),

$$\left\{ \frac{\partial^3 \mathbb{Q}}{\partial\tau\partial x^2}, \left[\begin{array}{l} m^2 \frac{\partial^3\mathbb{Q}}{\partial x^3} + m\left(\frac{\partial^2}{\partial\tau^2}(\frac{\partial \mathbb{Q}}{\partial\tau} + 2\tau\frac{\partial\mathbb{Q}}{\partial x}) - \frac{\partial^3(x\mathbb{Q})}{\partial\tau\partial x^2} + 2\left\{\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}, \frac{\partial^2\mathbb{Q}}{\partial x^2}\right\}_x\right) \\ + \frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}\left(\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}\frac{\partial^3\mathbb{Q}}{\partial x^3} - \frac{1}{2}\frac{\partial^3\mathbb{Q}}{\partial\tau^3}\right) + \left\{\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}, \frac{\partial^2(\tau\mathbb{Q})}{\partial\tau^2}\right\}_{x+\tau/2} \end{array}\right]\right\}_x$$

$$-\frac{1}{2}\left(\frac{\partial^3\mathbb{Q}}{\partial\tau\partial x^2}\right)^2\left(\frac{\partial^3\mathbb{Q}}{\partial\tau^3} - 4\frac{\partial^2\mathbb{Q}}{\partial\tau\partial x}\frac{\partial^3\mathbb{Q}}{\partial x^3}\right) = 0, \tag{1}$$

with “initial condition” :

$$\lim_{\tau\to-\infty}\mathbb{Q}(\tau,x) := \log\mathbb{P}(\sup\mathcal{A}(\tau) \le x) = -\int_x^\infty (\alpha - x)g^2(\alpha)d\alpha = \mathcal{F}(x).$$

Remember m is the number of outliers!

When $m \to \infty$, the m-Airy process tends to the Pearcey process:

$$\text{(m-Airy equation)}_{\left\{m=z^{-12},\ x=-\frac{3}{4}z^{-8}+\frac{t}{2}z^{-2}+yz, \tau=\frac{3}{2}z^{-4}+\frac{1}{2}z^{2}t\right\}}$$

$$= -\frac{8}{z^{26}}\left(\left\{\frac{\partial^3 \mathbb{Q}^{\mathcal{P}}}{\partial y^3}, \frac{\partial^3 \mathbb{Q}^{\mathcal{P}}}{\partial t^3} + \frac{1}{8}\left(\varepsilon_{_E} - 2t\frac{\partial}{\partial t} - 2\right)\partial_{_E}^2 \mathbb{Q}^{\mathcal{P}} - \frac{1}{2}\left\{\partial_{_E}^2 \mathbb{Q}^{\mathcal{P}}, \partial_{_E}\frac{\partial \mathbb{Q}^{\mathcal{P}}}{\partial t}\right\}_{\partial_{_E}}\right\}\right)$$

$$+O(\tfrac{1}{z^{23}})$$

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4. The Airy process $\mathcal{A}^{(m)}(t)$ with m wanderers leaving from distinct pts

Consider $n+m$ non-intersecting Brownian motions

$$x_{m+n}(t) < \ldots < x_1(t) \quad \text{with } -1 \leq t \leq 1 \text{ in } \mathbb{R}$$

and points $a_m \leq \ldots \leq a_1$ and $b_m \leq \ldots \leq b_1$ on $\mathbb{R}$.

$$x_i(-1) = a_i$$

$$x_i(1) = b_i \qquad \text{for } 1 \leq i \leq m$$

$$x_{m+1}(\pm 1) = \ldots = x_{m+n}(\pm 1) = 0$$

4. The Airy process $\mathcal{A}^{(m)}(t)$ with m wanderers leaving from distinct pts

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and points $a_m \leq \ldots \leq a_1$ and $b_m \leq \ldots \leq b_1$ on $\mathbb{R}$.

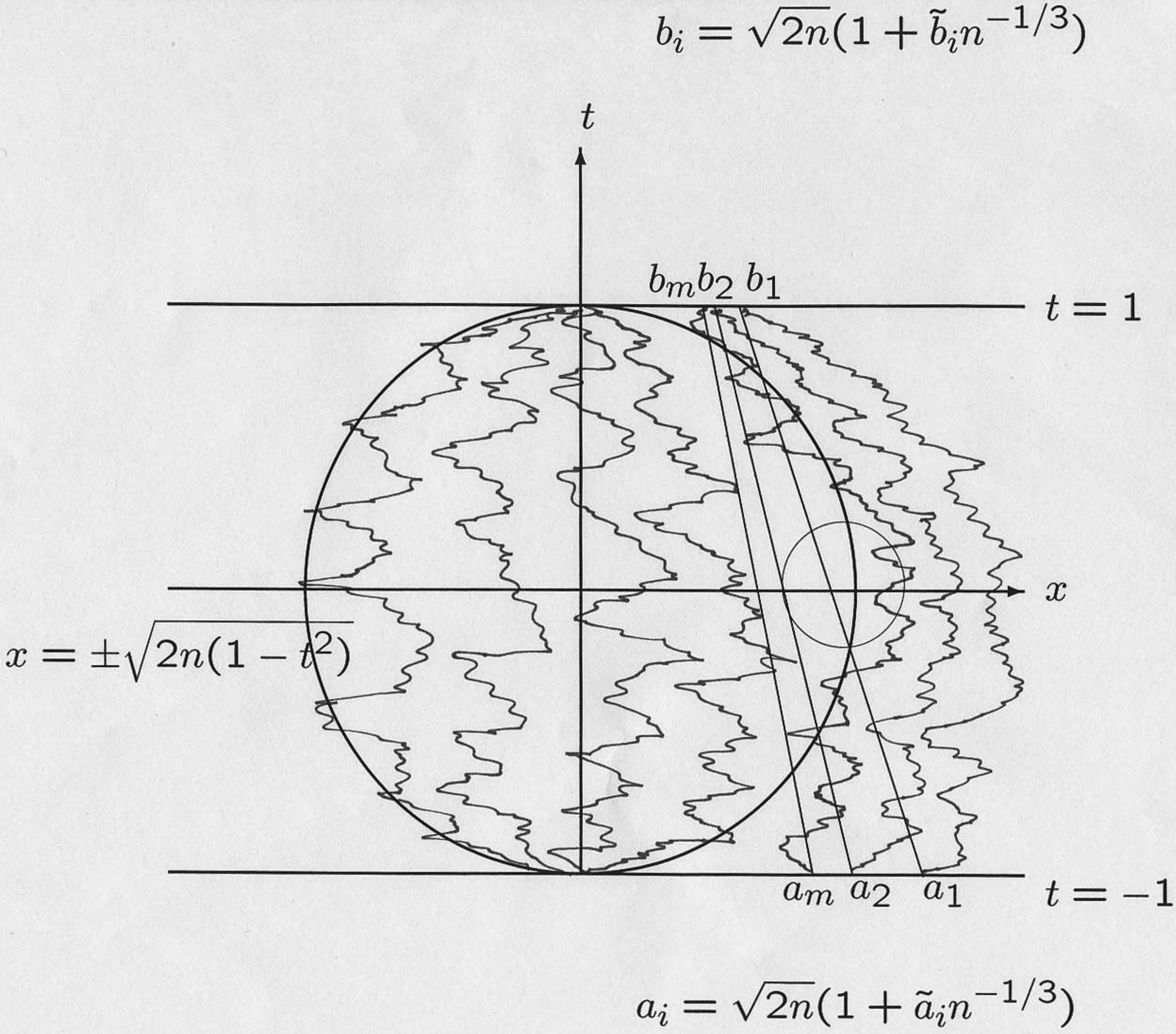
$$x_i(-1) = a_i = \sqrt{2n}\left(1 + \frac{\tilde{a}_i}{n^{1/3}}\right)$$

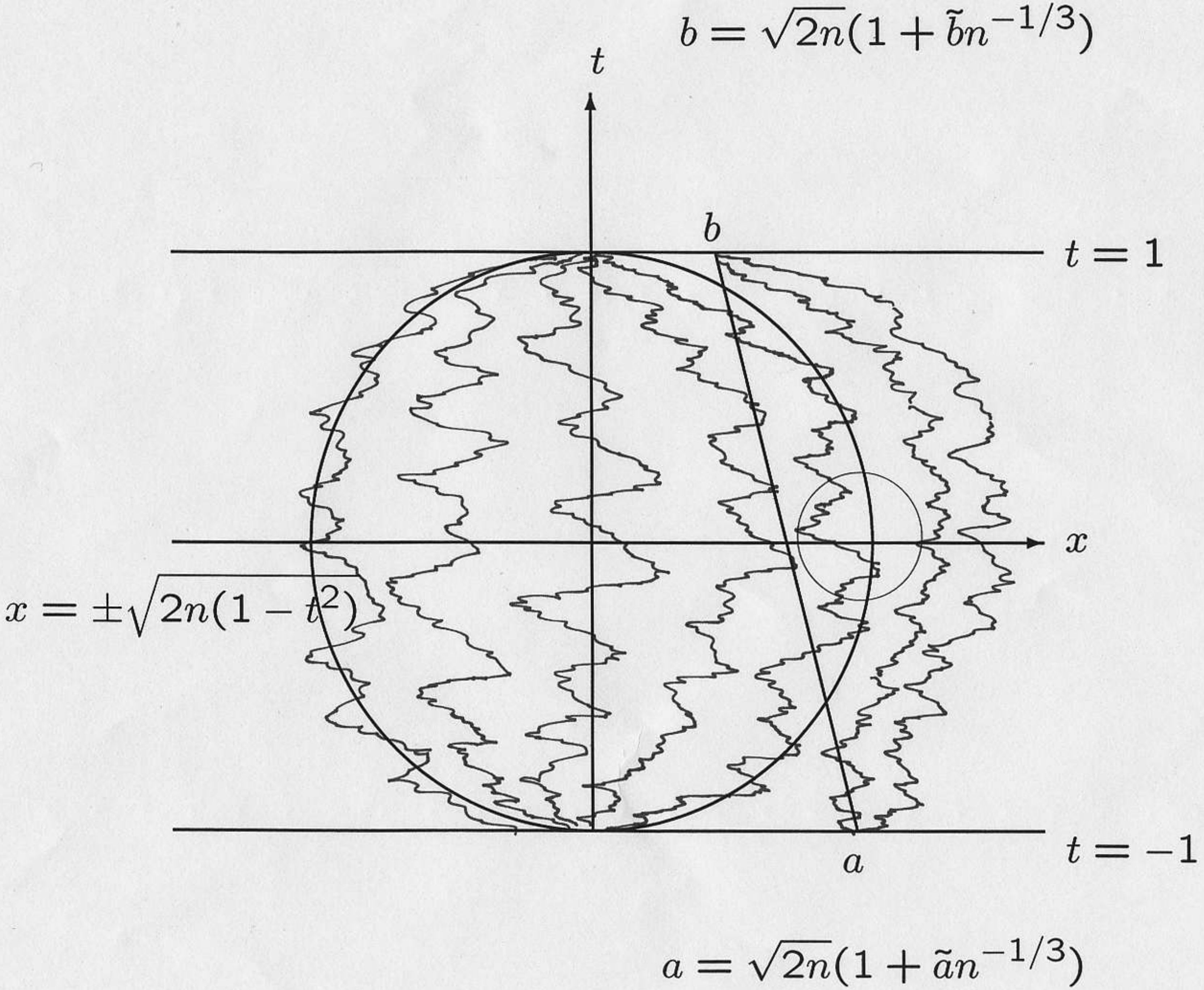
$$x_i(1) = b_i = \sqrt{2n}\left(1 - \frac{\tilde{b}_i}{n^{1/3}}\right), \qquad \text{for } 1 \leq i \leq m$$

$$x_{m+1}(\pm 1) = \ldots = x_{m+n}(\pm 1) = 0$$

with

$$\tilde{a}_m < \ldots < \tilde{a}_1 < \tilde{b}_1 < \ldots < \tilde{b}_m$$


$b_i = \sqrt{2n}(1 + \tilde{b}_i n^{-1/3})$
t
$b_m b_2\ b_1$
$t = 1$
x
$x = \pm\sqrt{2n(1-t^2)}$
$a_m\ a_2\ a_1$
$t = -1$
$a_i = \sqrt{2n}(1 + \tilde{a}_i n^{-1/3})$


$b = \sqrt{2n}(1 + \tilde{b} n^{-1/3})$
t
b
$t = 1$
x
$x = \pm\sqrt{2n(1-t^2)}$
$t = -1$
a
$a = \sqrt{2n}(1 + \tilde{a} n^{-1/3})$

Look through the "Airy" microscope at **the fluctuations of the paths** about the point $(x,t)=(\sqrt{2n},0)$ on the curve $x=\sqrt{2n(1-t^2)}$, for very large n: **New Airy process with** m**-outliers** $\mathcal{A}_m^{(a,b)}(\tau)$:

Pick times $\tau_1,\ \tau_2,\dots,\tau_\ell$ and intervals $E_1,\dots,E_\ell\subset\mathbb{R}$. Then,

$$\begin{aligned}\lim_{n\to\infty}\mathbb{P}_{\mathrm{ab}}&\left(\bigcap_{k=1}^{\ell}\left\{\text{all } x_i\left(\frac{\tau_k}{n^{1/3}}\right)\in\sqrt{2n}+\frac{E_k^c-\tau_k^2}{\sqrt{2}n^{1/6}}\right\}\right)\\ =\ &\det(\mathbb{I}-\chi_E K_m^{\tilde a,\tilde b}\chi_E)=:\mathbb{P}\left(\bigcap_{k=1}^{\ell}\left\{\mathcal{A}_m^{(\tilde a,\tilde b)}(\tau_k)\cap E_k=\emptyset\right\}\right)\end{aligned}$$

with (matrix Fredholm determinant) (all $\tilde a_k<$ all $\tilde b_k$)

$$\begin{aligned}K_m^{\tilde a,\tilde b}(\tau_i,\xi_i;\tau_j,\xi_j)=&-\frac{\mathbb{I}(\tau_j>\tau_i)}{\sqrt{4\pi(\tau_j-\tau_i)}}e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_j-\tau_i)}-\frac{1}{2}(\tau_j-\tau_i)(\xi_j+\xi_i)+\frac{1}{12}(\tau_j-\tau_i)^3}\\ &+\frac{1}{(2\pi i)^2}\int_\Gamma dU\int_{\Gamma'}dV\,\frac{e^{-\frac{U^3}{3}+\xi_jU}}{e^{-\frac{V^3}{3}+\xi_iV}}\ \frac{\prod_{k=1}^m\left(\frac{V-\tilde a_k+\tau_i}{U-\tilde a_k+\tau_j}\right)\left(\frac{U-\tilde b_k+\tau_j}{V-\tilde b_k+\tau_i}\right)}{(U+\tau_j)-(V+\tau_i)}.\end{aligned}$$

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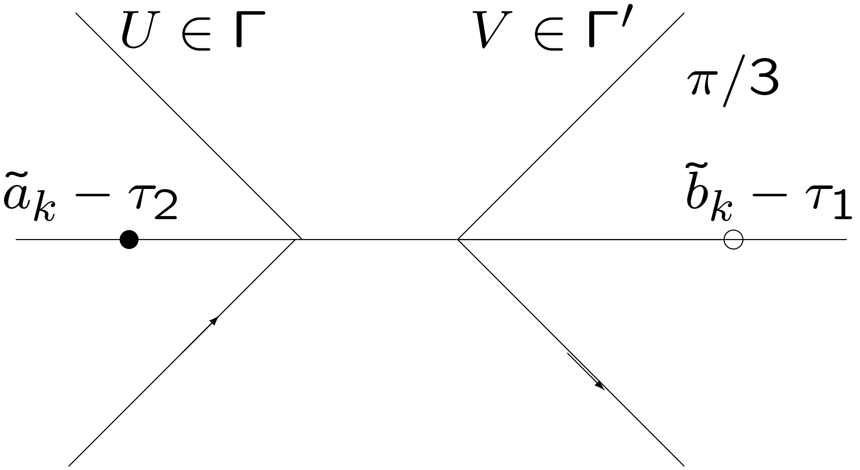
where $\Gamma_{\tilde{a}-\tau_j>}$ and $\Gamma_{<\tilde{b}-\tau_i}$ denote the contours below.



$$\tilde{a}_m < \ldots < \tilde{a}_1 < \tilde{b}_1 < \ldots < \tilde{b}_m \quad \text{and} \quad \tau_1 \leq \tau_2$$

(holds also in the case that all $\tilde{a}_m = \ldots = \tilde{a}_1 < \tilde{b}_1 = \ldots = \tilde{b}_m$).

- Kernel also found by Borodin-Péché '08 in the context of a directed percolation model

5. Limit to the Pearcey process

Airy process $\mathcal{A}_m^{(\tilde{a},\tilde{b})}(t)$, with m-outliers, obtained by letting $n \to \infty$,

$$a = \sqrt{2n}\left(1 + \frac{\tilde{a}}{n^{1/3}}\right) \quad \text{and} \quad b = \sqrt{2n}\left(1 - \frac{\tilde{b}}{n^{1/3}}\right), \quad \text{with } \tilde{a} < \tilde{b} \qquad (2)$$

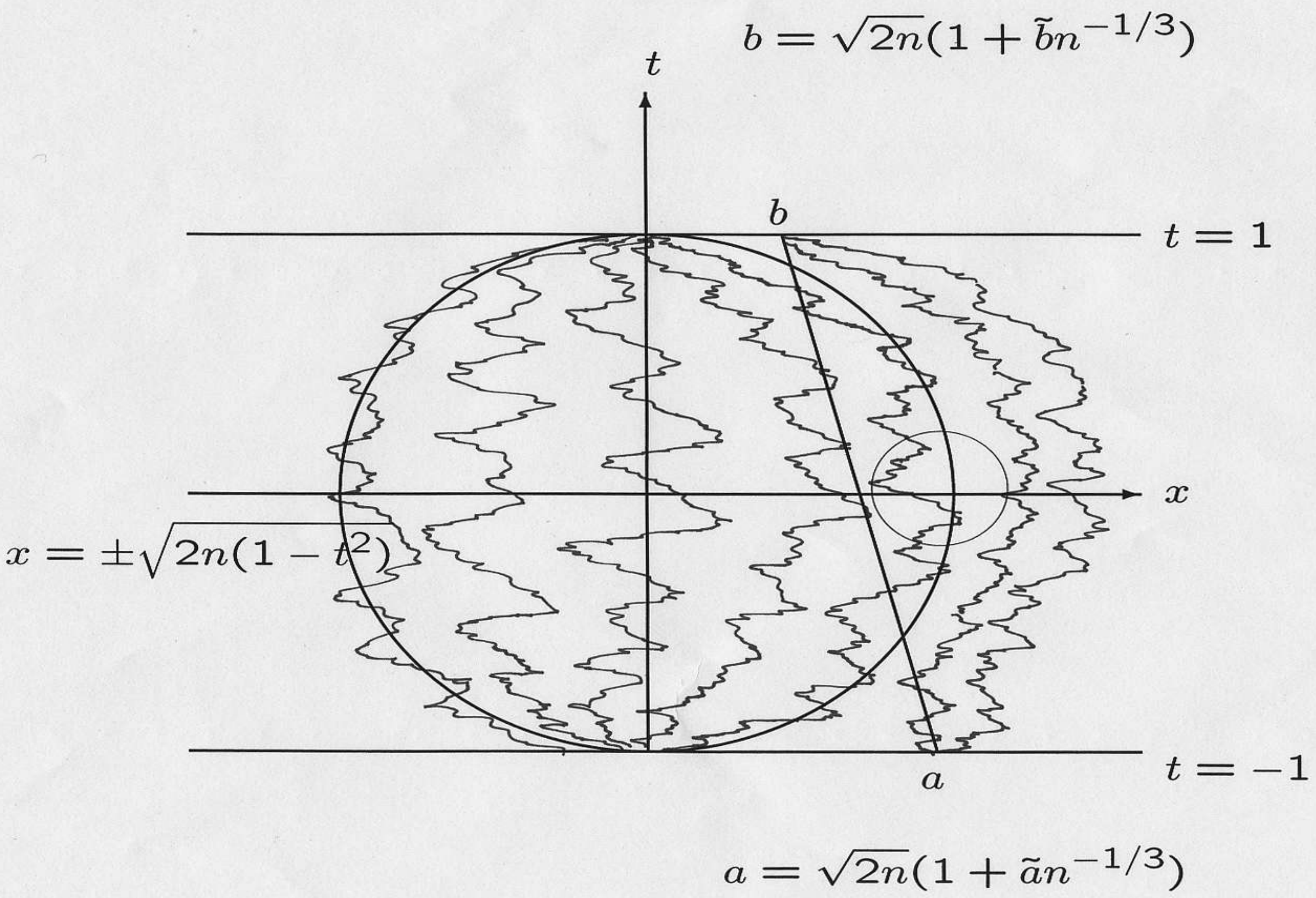
Further rescaling, and then letting $m \to \infty$:

$$\tilde{a} = \alpha m^{1/3},\ \tilde{b} = \beta m^{1/3}, \quad \tau = T m^{1/3}, \quad \xi = X m^{2/3}$$

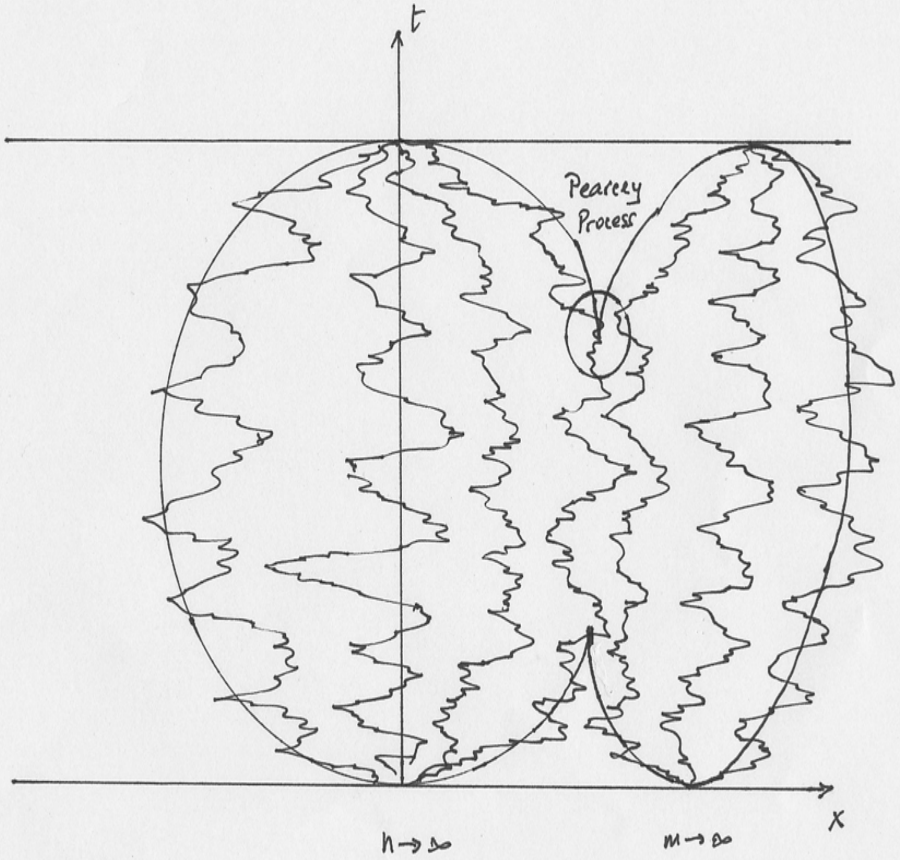
Remember the fluctuations near $(x, t) = (\sqrt{2n}, 0)$ for m finite:

$$K_m^{\tilde{a},\tilde{b}}(\tau_i, \xi_i; \tau_j, \xi_j) = -\frac{\mathbb{I}(\tau_j > \tau_i)}{\sqrt{4\pi(\tau_j - \tau_i)}} e^{-\frac{(\xi_2 - \xi_1)^2}{4(\tau_j - \tau_i)} - \frac{1}{2}(\tau_j - \tau_i)(\xi_j + \xi_i) + \frac{1}{12}(\tau_j - \tau_i)^3}$$

$$+ \frac{1}{(2\pi i)^2} \int_{\Gamma_{\tilde{a}-\tau_j >}} dU \int_{\Gamma_{< \tilde{b}-\tau_i}} dV \, \frac{e^{-U^3/3 + \xi_j U}}{e^{-V^3/3 + \xi_i V}} \, \frac{\prod_{k=1}^m \left(\frac{V - \tilde{a}_k + \tau_i}{U - \tilde{a}_k + \tau_j}\right)\left(\frac{U - \tilde{b}_k + \tau_j}{V - \tilde{b}_k + \tau_i}\right)}{(U + \tau_j) - (V + \tau_i)}.$$



m outliers: $\tilde{a}=\alpha m^{1/3}$, $\tilde{b}=\beta m^{1/3}$, space-time $\underbrace{\tau = T_{\pm} m^{1/3}, \quad \xi = X m^{2/3}}_{\text{(location of tip of cusp in next slide)}}$
with $m \to \infty$



$$\tau = T_{\pm} m^{1/3}, \quad \xi = X m^{2/3} : \text{ location of tips of cusps}$$

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Airy process $\mathcal{A}_m^{(\tilde{a},\tilde{b})}(t)$, with m-outliers, obtained by letting $n \to \infty$,

$$a = \sqrt{2n}\left(1 + \frac{\tilde{a}}{n^{1/3}}\right) \quad \text{and} \quad b = \sqrt{2n}\left(1 - \frac{\tilde{b}}{n^{1/3}}\right), \quad \text{with } \tilde{a} < \tilde{b}$$

Further rescaling, and then letting $m \to \infty$:

$$\tilde{a} = \alpha m^{1/3},\ \tilde{b} = \beta m^{1/3}, \quad \tau = T m^{1/3}, \quad \xi = X m^{2/3},\ \text{with} \ \ \alpha < \beta$$

Airy process $\mathcal{A}_m^{(\tilde a,\tilde b)}(t)$, with m-outliers, obtained by letting $n \to \infty$,

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Further rescaling, and then letting $m \to \infty$:

$$\tilde a = \alpha m^{1/3},\ \tilde b = \beta m^{1/3}, \quad \tau = T m^{1/3}, \quad \xi = X m^{2/3}, \text{ with } \ \alpha < \beta$$

Then, given $\alpha < \beta$, the equations,

$$\begin{cases} \beta - \alpha = \dfrac{4\sigma^4 x^3}{2 - x} \\ \\ 4\sigma^6 x^4 - 2x + 3 = 0, \text{(elliptic curve)} \end{cases}$$

Airy process $\mathcal{A}_m^{(\tilde a,\tilde b)}(t)$, with m-outliers, obtained by letting $n\to\infty$,

$$a=\sqrt{2n}\left(1+\frac{\tilde a}{n^{1/3}}\right)\quad\text{and}\quad b=\sqrt{2n}\left(1-\frac{\tilde b}{n^{1/3}}\right),\quad\text{with }\tilde a<\tilde b$$

Further rescaling, and then letting $m\to\infty$:

$$\tilde a=\alpha m^{1/3},\ \tilde b=\beta m^{1/3},\quad \tau=Tm^{1/3},\quad \xi=Xm^{2/3},\ \text{with}\ \ \alpha<\beta$$

Then, given $\alpha<\beta$, the equations,

$$\begin{cases}\beta-\alpha=\dfrac{4\sigma^4x^3}{2-x}\\[2ex] 4\sigma^6x^4-2x+3=0,\text{(elliptic curve)}\end{cases}$$

have a

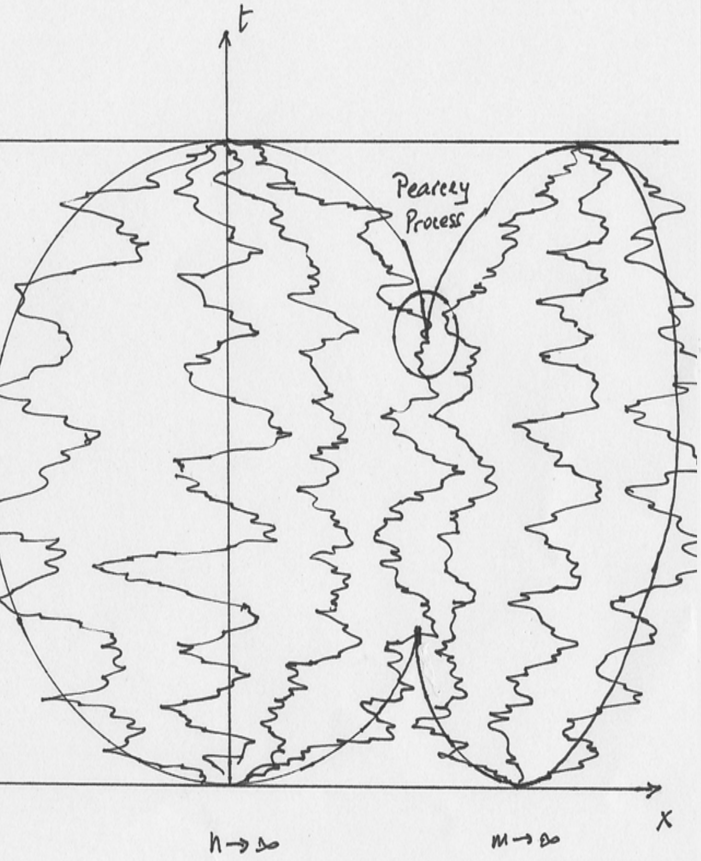
unique solution $:=$ $(x,\sigma_+)\in\left((\frac32,2)\times(-\frac12,0)\right)$ (**opening cusp:** (X,T_+))

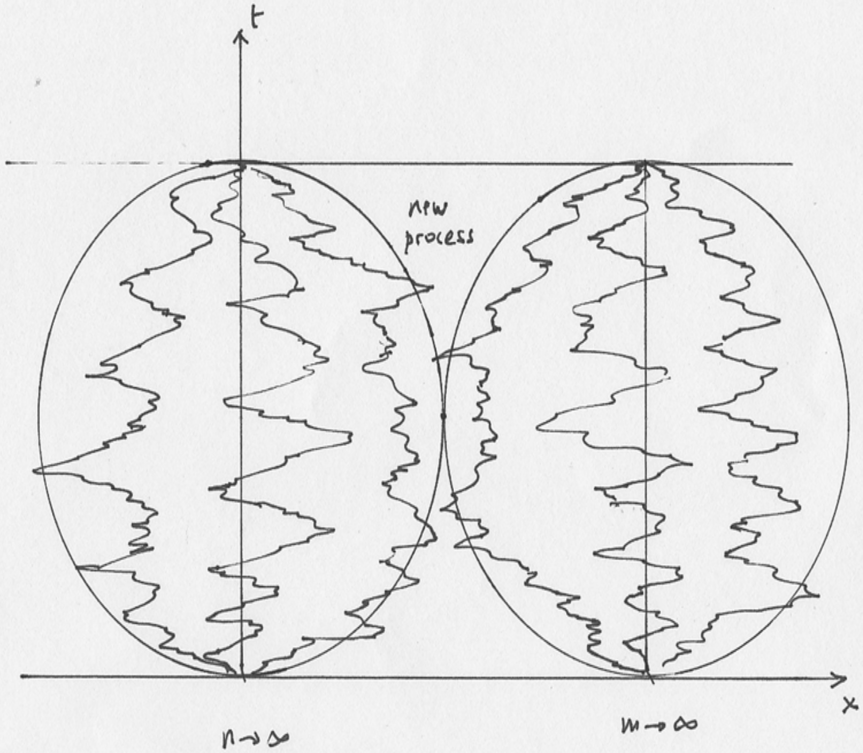
unique solution $:=$ $(x,\sigma_-)\in\left((\frac32,2)\times(0,\frac12)\right)$ (**closing cusp:** (X,T_-)).

Two (id.) Pearcey processes $\mathcal{P}(\theta)$ about two cusps, $(T_-<\frac{\alpha+\beta}{2}<T_+)$

$$\tau\sim T_\pm m^{1/3},\quad \xi\sim Xm^{2/3},\qquad\text{with}\ \ T_\pm:=\frac{\alpha+\beta}{2}-\frac{2\sigma_\pm}{2-x},\quad X:=\sigma_\pm^2(1-2x),$$

6. Limit to a new process $\mathrm{Q}(t)$





Letting the two Pearcey cusps approach each other:

Quintic kernel $= K^{\mathcal{Q}}(\theta; v_1, v_2)$

$$= \frac{1}{(2\pi i)^2} \int\int \frac{dV\,dU}{V-U} \quad \frac{e^{-\frac{2}{5}U^5+\frac{1}{3}\theta U^3 - Uv_1}}{e^{-\frac{2}{5}V^5+\frac{1}{3}\theta V^3 - Vv_2}}.$$

$$K_m^{\tilde{a},\tilde{b}}(\tau_i,\xi_i;\tau_j,\xi_j) = -\frac{\mathbb{I}(\tau_j>\tau_i)}{\sqrt{4\pi(\tau_j-\tau_i)}} e^{-\frac{(\xi_2-\xi_1)^2}{4(\tau_j-\tau_i)}-\frac{1}{2}(\tau_j-\tau_i)(\xi_j+\xi_i)+\frac{1}{12}(\tau_j-\tau_i)^3}$$

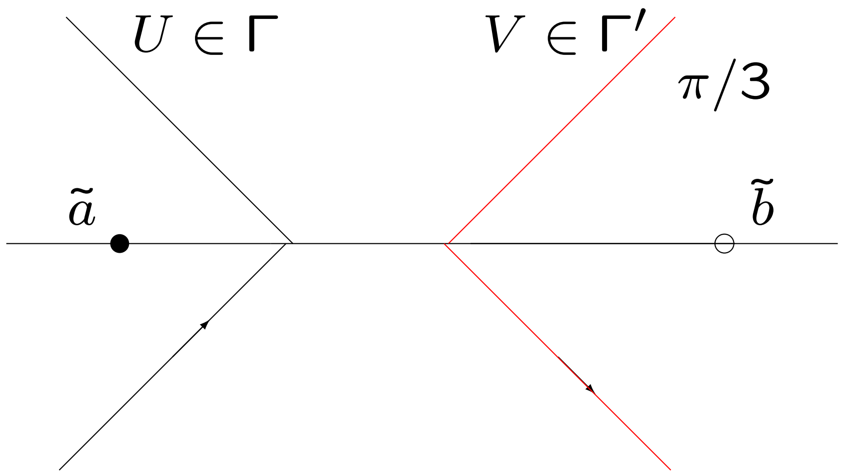
$$+\frac{1}{(2\pi i)^2}\int_{\Gamma_{\tilde{a}-\tau_j>}} dU \int_{\Gamma_{<\tilde{b}-\tau_i}} dV \, \frac{e^{-U^3/3+\xi_j U}}{e^{-V^3/3+\xi_i V}} \; \frac{\left(\frac{V-\tilde{a}+\tau_i}{U-\tilde{a}+\tau_j}\right)^m \left(\frac{U-\tilde{b}+\tau_j}{V-\tilde{b}+\tau_i}\right)^m}{(U+\tau_j)-(V+\tau_i)}.$$

for $\tilde{a} < \tilde{b}$

- Analytically continue this kernel, by letting $\tilde{a}$ and $\tilde{b}$ move around in the complex plane, such that $\tilde{b} < \tilde{a}$.

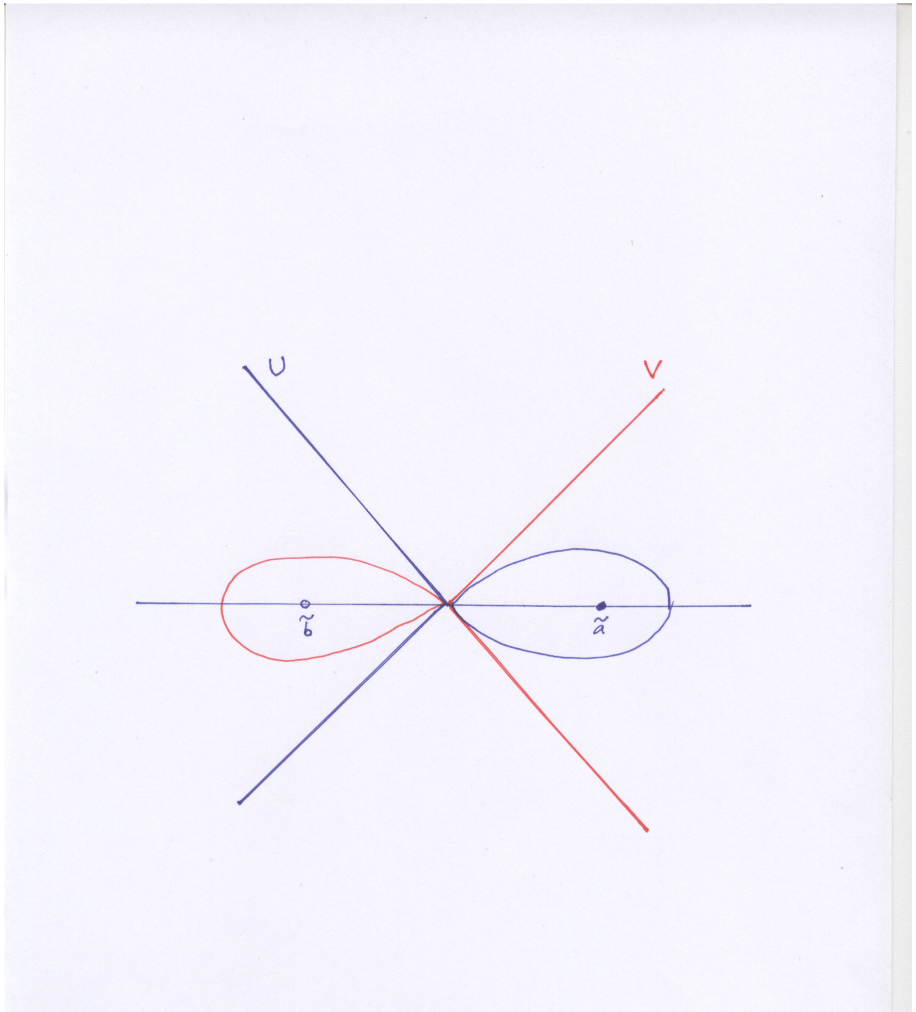
- Obtain the quintic kernel, by taking limit $m \to \infty$.

$\tilde{a} < \tilde{b}$



$\Downarrow$

$\tilde{b} < \tilde{a}$



Thank you!