Yad Hashmona, March 29th , 2009

From Random Matrices to Evaporating Droplets

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Outline

Random matrices

Viscous fingering

Dispersive regularization

The “gortex model”

Evaporating droplets

The selection problem

Normal Random Matrices

M: $N \times N$ *a complex matrix such that* $[M, M^\dagger] = 0$

$$\tau_N = \int d\mu(M)\, e^{-\frac{1}{\hbar}\operatorname{Tr} W\left(M, M^\dagger\right)}$$

$$W(M, M^\dagger) = MM^\dagger - \sum_{k=1}^{\infty} \left(t_k M + C.c\right)$$

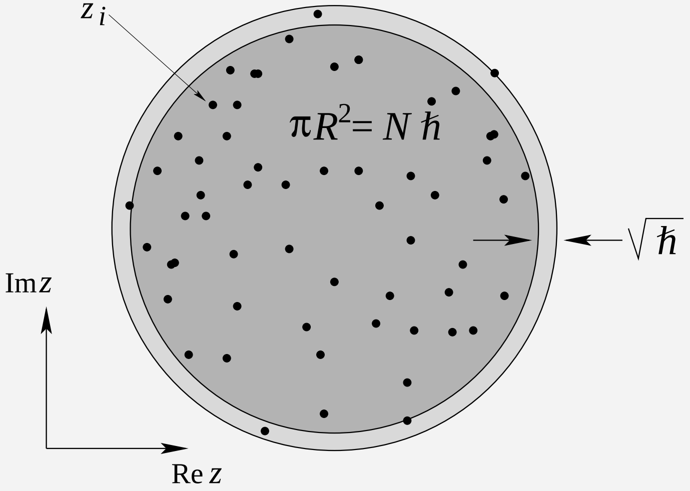
$$\tau_N = \int \prod_{i=1}^{N} d^2 z_i \prod_{i>j} \left|z_i - z_j\right|^2 e^{-\frac{1}{\hbar}\sum_{i=1}^{N} W(z_i, \overline{z}_i)}$$

Droplets of eigenvalues

$$\tau_N = \int \prod_{i=1}^{N} d^2 z_i \prod_{i>j} \left| z_i - z_j \right|^2 \exp\left\{ -\frac{1}{\hbar} \sum_{i=1}^{N} \left[\left| z_i \right|^2 - \sum_{k=1}^{\infty} \left(t_k z_i + C.c \right) \right] \right\}$$

Example:

$$t_k = 0 \quad \forall k$$



$$\left\langle \rho(z) \right\rangle = \left\langle \sum_{i=1}^{N} \delta\left(z - z_i \right) \right\rangle$$

Saddle point approximation

$$\tau_N = \int D\rho(z)\exp\left(-\frac{1}{\hbar}S[\rho]\right) \qquad \rho(z) = \sum_{i=1}^{N}\delta\left(z - z_i\right)$$

$$S[\rho] = \int d^2 z\left(W(z,\overline{z})\,\rho(z) - \hbar\int d^2 z'\rho\left(z'\right)\ln\left|z - z'\right|^2 \rho(z)\right)$$

$$\frac{\delta S}{\delta\rho} = 0 \qquad \Rightarrow \qquad W(z,\overline{z}) - \hbar\int d^2 z'\rho\left(z'\right)\ln\left|z - z'\right|^2 = 0$$

$$\rho(z) = \begin{cases} \dfrac{1}{\pi\hbar} & \text{within the droplet} \\ 0 & \text{outside the droplet} \end{cases}$$

The dispersionless limit

$$N \to \infty, \quad \hbar \to 0, \quad N\hbar = t_0$$

$$\int d^2z P_n(z) P_m(\overline{z}) e^{-\frac{1}{\hbar}W(z,\overline{z})} \propto \delta_{mn}$$

$$\psi_N(z) = P_N(z) e^{-\frac{1}{\hbar}\left(\frac{|z|^2}{2} - \sum_k t_k z^k\right)}$$

$$\psi(z) \sim \exp\left[-\frac{1}{\hbar}\left(\frac{|z|^2}{2} - \int dz S(z)\right)\right]$$

$$F(x,y) = 0$$

$$F\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) = 0$$

$$\frac{\partial}{\partial z}\left[\frac{1}{\hbar}\left(\frac{|z|^2}{2} - \int dz S(z)\right)\right] \quad \Rightarrow \quad \overline{z} = S(z)$$

The Schwarz function

Viscous Fingering:

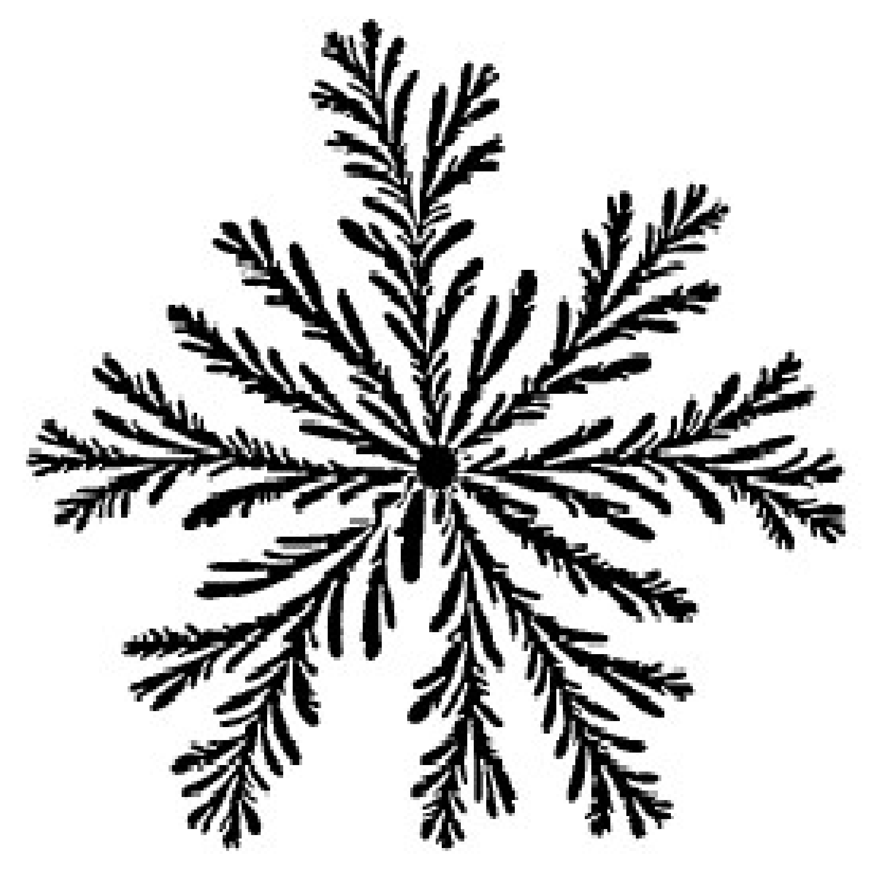
The Saffman-Taylor problem

$$\nabla^2 P = 0 \quad \text{in} \quad D_-$$

$$P = 0 \quad \text{in} \quad D_+$$

$$P\big|_{z\in\partial D} = \gamma\kappa$$

$$v_n = -\partial_n P$$



The idealized Saffman-Taylor problem

(zero surface tension)

$$P\big|_{z\in\partial D} = 0$$

Integrable:

$$t_k = -\frac{1}{\pi k}\iint_{\mathcal{D}_-} d^2z \; z^{-k}, \qquad\qquad z = x + iy.$$

The external harmonic moments are constants of motion.

(independent of the droplet's area)

Relation to Random matrices

$$W(z,\bar{z}) = \frac{1}{\pi}\int_{D^+} d^2z' \log\left|z - z'\right|^2$$

The potential generated by positive charge of constant uniform density in D^+

$$W(z,\bar{z}) = |z|^2 - \frac{1}{\pi}\int_{D^-} d^2z' \log\left|z - z'\right|^2$$

$$= |z|^2 + 2\,\mathrm{Re}\int_{D^-} d^2z' \sum_k \frac{1}{\pi k}\left(\frac{z}{z'}\right)^k$$

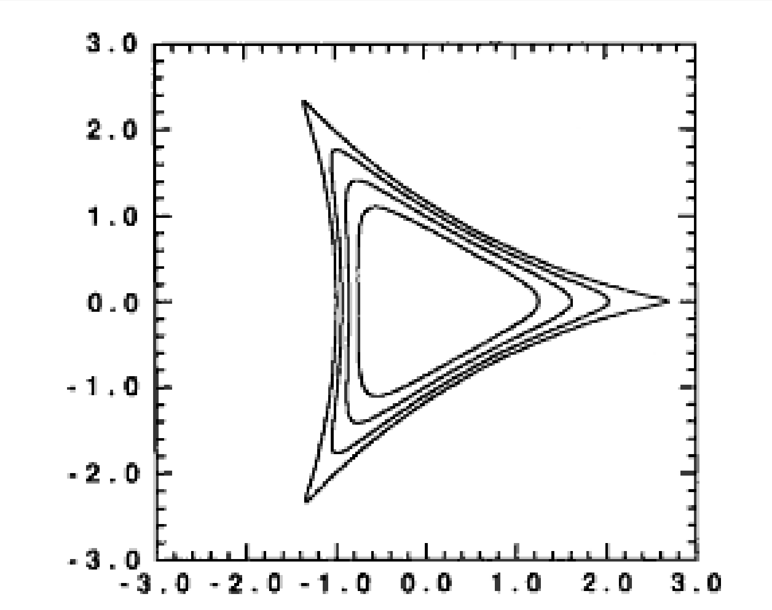
$$= |z|^2 - \left(\sum_k t_k z^k + C.c\right)$$

Singularities

$t_3 \neq 0$

A boundary singularity appears, generically, after finite time.

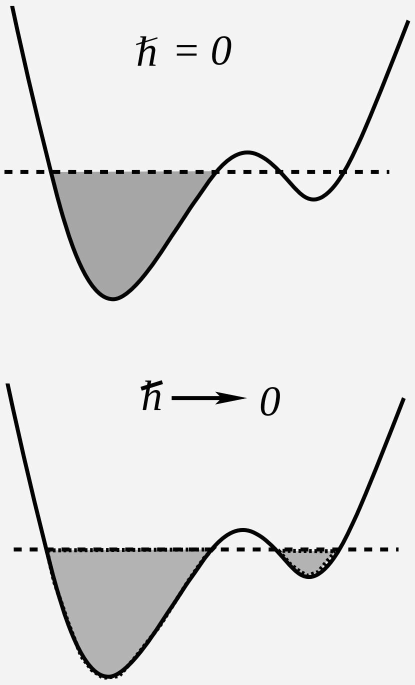
$$t_0 \to t_0^*$$

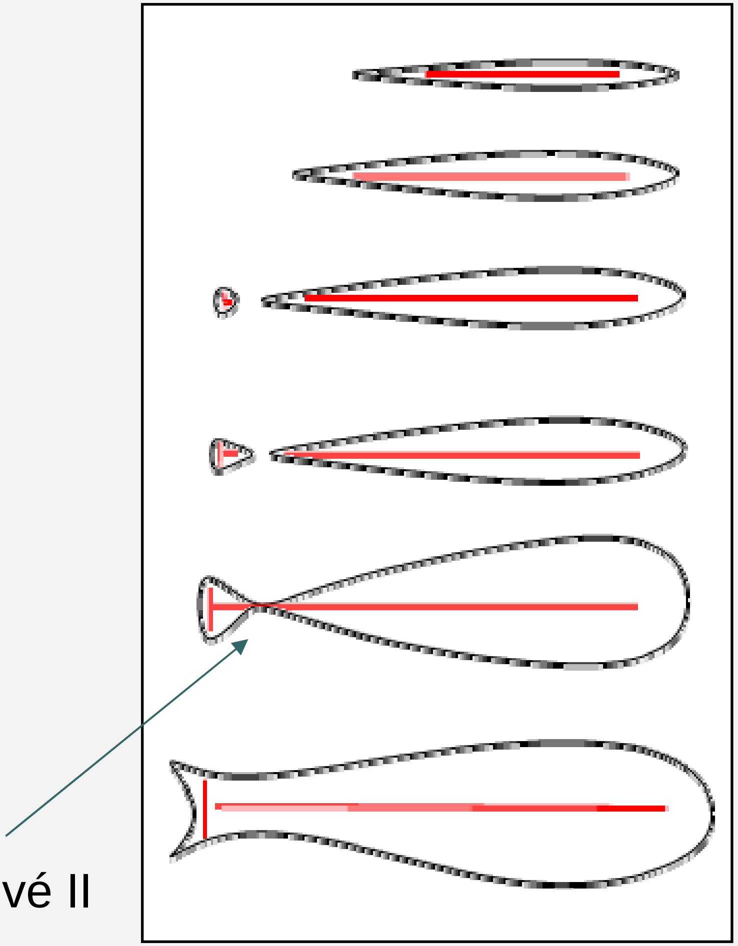


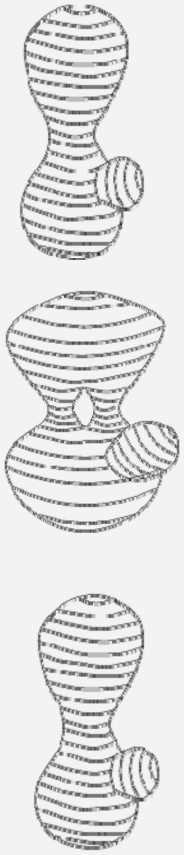
Surface tension hinders the singularity but also destroys the integrable structure.

Dispersive regularization

Amounts for the creation of new bubbles having equal pressure.

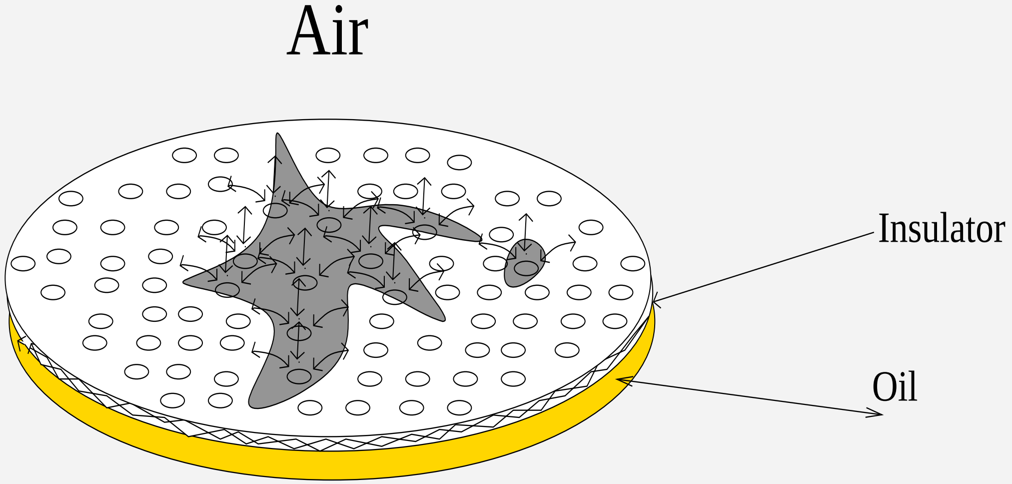






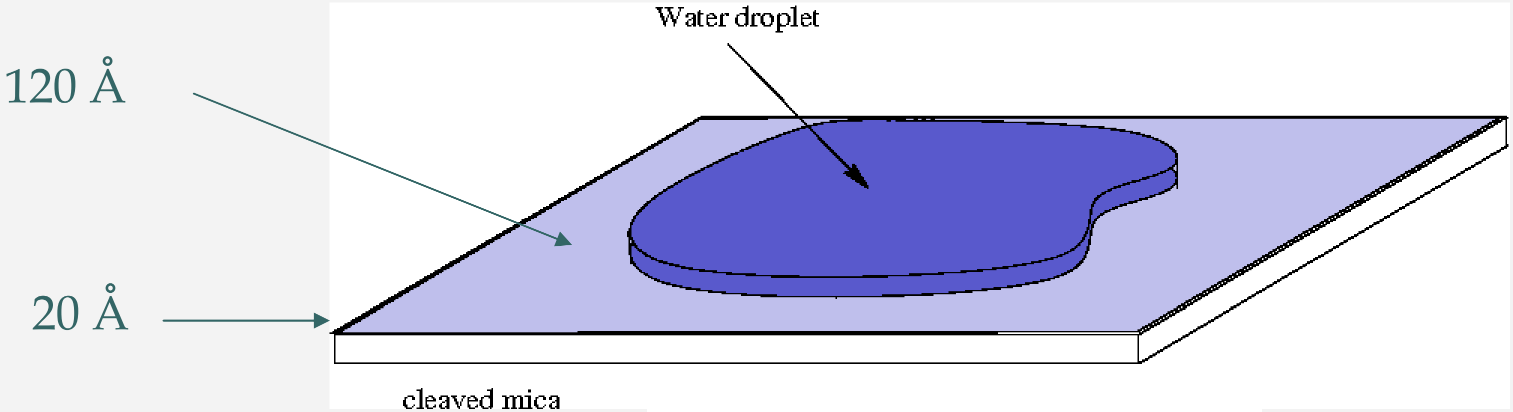
(Lee, Bettelheim & Wiegmann)

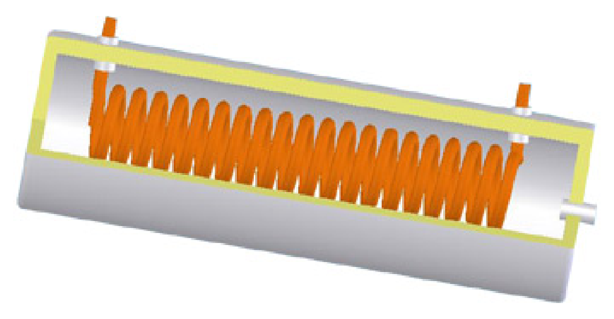
The gotrex Hele-Shaw model



Volatile thin water films on mica:

S.G. Lipson et al.

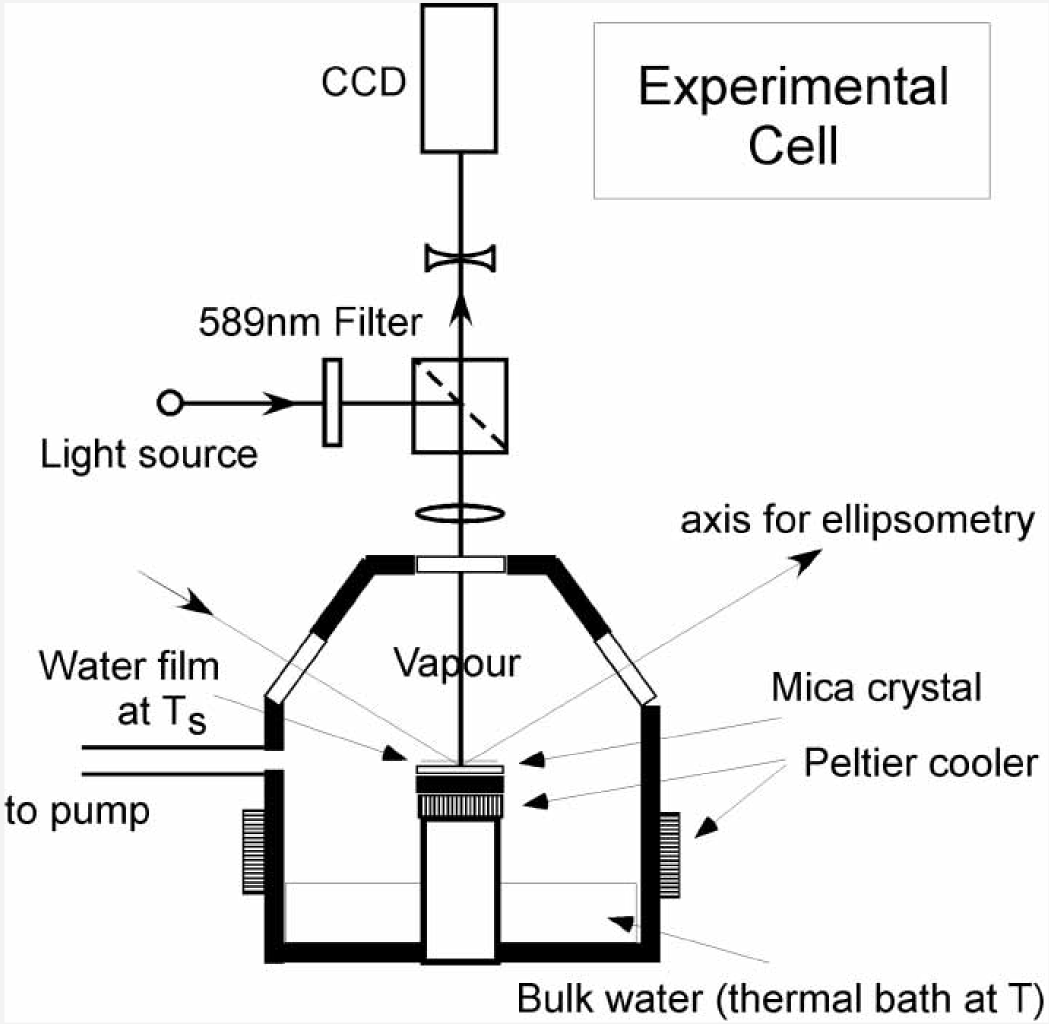




First Order phase transition between two film thicknesses resulting from competition between van der Waals and polar surface forces

The Experimental Apparatus:

Interference microscopy



Equations of motion

Assumptions:

1. *Fluid temperature and density are constant.*
2. *The contact line is not pinned.*
3. *Lubrication approximation: Film thickness is sufficiently small so that inertia terms may be neglected.*
4. *Effects of gravity are neglected .*
5. *Evaporation is uniform within the droplet.*

Equations of motion (cont.)

Within the droplet (away from the contact line):

$$\vec{v} = -\vec{\nabla} P$$

Darcy's law

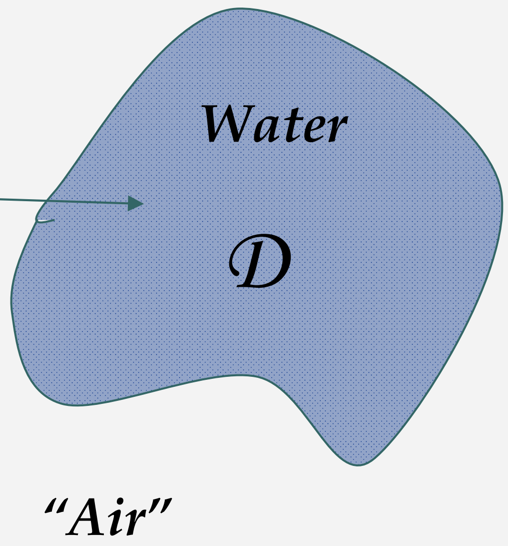
$$\vec{\nabla} \cdot \vec{v} = -\beta$$

constant evaporation rate

Outside the droplet (away from the contact line):

$$P = 0$$

Equations of motion of the idealized system (zero surface tension)



$$\vec{\nabla}^2 P = \beta$$

$$P = 0$$

$$P(z) = 0 \qquad \text{for} \qquad z \in \partial\mathcal{D}$$

Integrability

(Entov & Étingof, 1992)

Internal harmonic moments:

$$t_k = \frac{1}{\pi}\iint_{\mathcal{D}} d^2z \ z^k, \qquad z = x + iy.$$

$$t_k(t) = t_k(0)\, e^{-\beta t}$$

Proof:

$$\frac{dt_k}{dt} = \frac{1}{\pi}\oint_{\partial\mathcal{D}} dl \ z^k v_n, \qquad v_n = -\frac{\partial P}{\partial n}$$

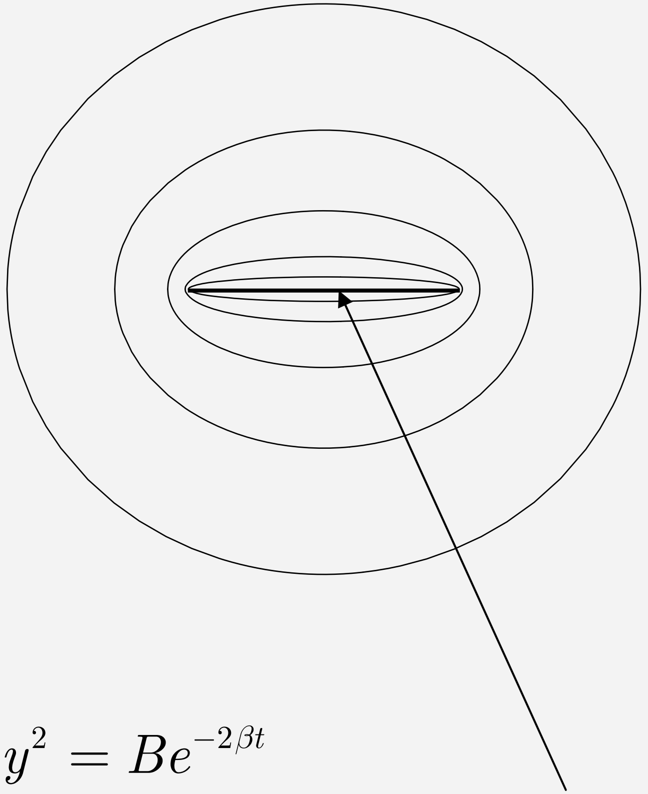
$$\frac{dt_k}{dt} = \frac{1}{\pi}\oint_{\partial\mathcal{D}} dl \ \left(P\partial_n z^k - z^k\partial_n P\right) = \frac{1}{\pi}\iint_{\mathcal{D}} d^2z \ \Big(P\underset{\substack{\uparrow\\ 0}}{\nabla^2 z^k} - z^k\underset{\substack{\uparrow\\ \beta}}{\nabla^2 P}\Big)$$

$$= -\beta t_k$$

Example: evaporation of elliptical droplets

$$P = \mathrm{Re}\left(\frac{\beta}{4}|z|^2 - \frac{\alpha(t)}{4}z^2 - \gamma(t)\right)$$

$$= \frac{\beta - \alpha(t)}{4}x^2 + \frac{\beta + \alpha(t)}{4}y^2 - \gamma(t)$$



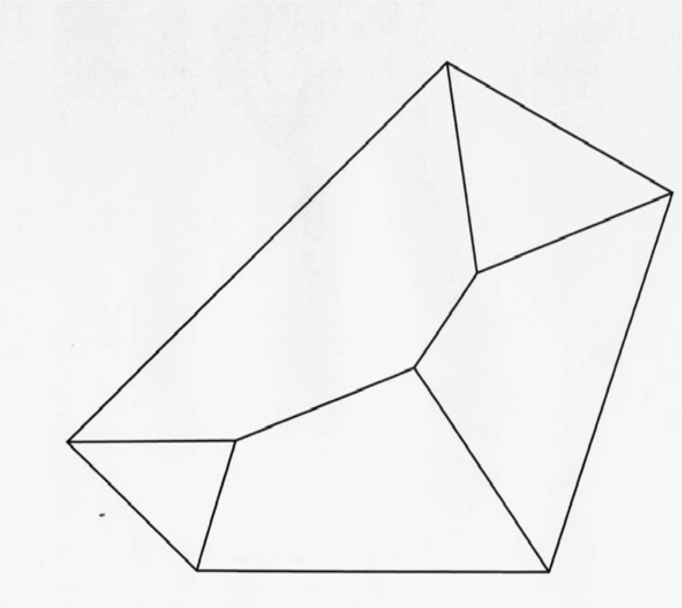
The boundary condition, P=0, is, clearly, the equation of an ellipse:

$$\left(\sqrt{1 + Ae^{-2\beta t}} - 1\right)x^2 + \left(\sqrt{1 + Ae^{-2\beta t}} + 1\right)y^2 = Be^{-2\beta t}$$

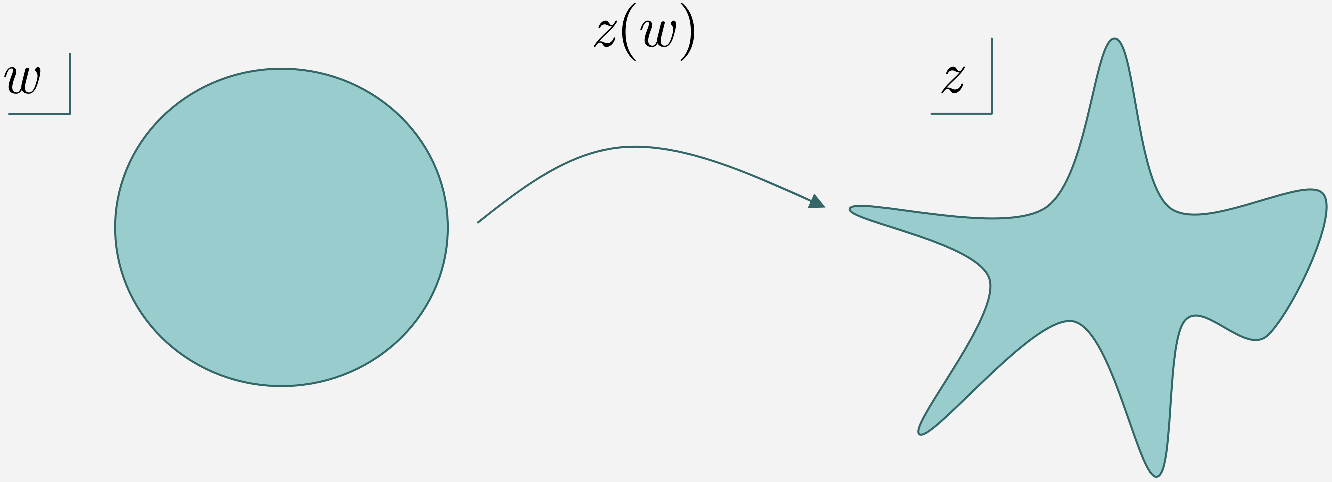
"mother body"

“Mother bodies” *(Gustafsson, 2004)*

A convex polyhedron:



Genus-zero solutions



$$S(z) = \overline{z}\left(\frac{1}{w(z)}\right)$$

$$t_k = \frac{1}{2\pi i}\oint_{\partial D} dl S(z)\, z^k$$

Example: The Joukowski map

$$z(w) = r(t)\omega + \frac{u_0(t)}{w - a(t)} + u_1(t), \qquad a(t) > 1$$

The Schwarz function:

$$\left.\begin{aligned} S(z) &= \bar{z}\left(\frac{1}{w(z)}\right) = \frac{r}{w(z)} + \frac{u_0}{\frac{1}{w(z)} - a} + u_1 \\ S(z) &\sim \frac{\mu_1}{z} + \frac{\mu_2}{z - q} \end{aligned}\right\} \quad \begin{aligned} & z(0) = 0, \quad z\left(\tfrac{1}{a}\right) = q, \\ & \mu_1 = r\left.\frac{dz}{dw}\right|_{w=0}, \quad \mu_2 = -\frac{u_0}{a^2}\left.\frac{dz}{dw}\right|_{w=1/a} \end{aligned}$$

$$t_k = \sum_j \mu_j q_j^k = t_k(0)e^{-\beta t}$$

$$\mu_1 = \mu_1(0)e^{-\beta t}, \quad \mu_2 = \mu_2(0)e^{-\beta t}, \qquad q = const$$

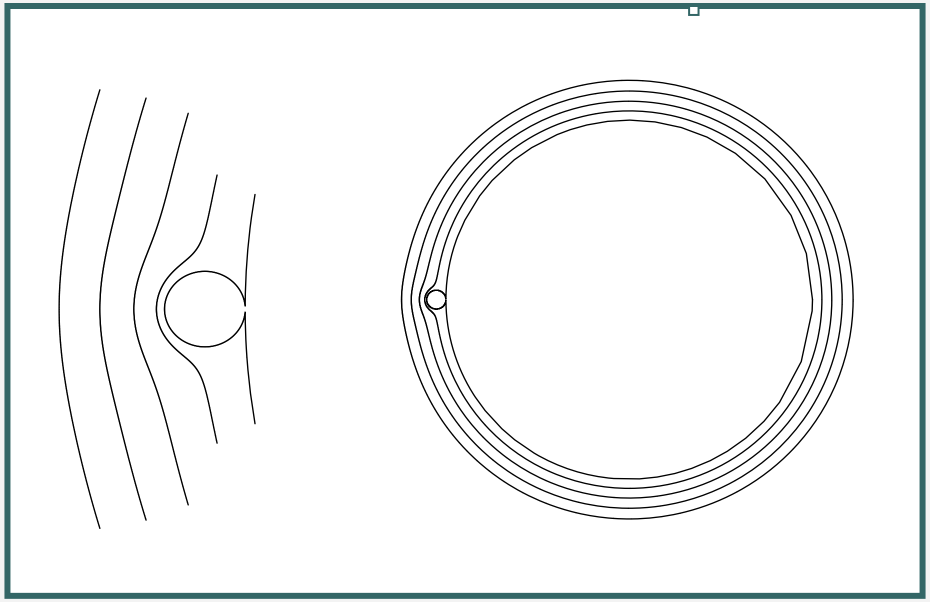
The Joukowski map (cont.)

$$-\frac{u_0}{a} + u_1 = 0$$

$$\frac{r}{a} + \frac{u_0}{\frac{1}{a} - a} + u_1 = q$$

$$\left(r - \frac{u_0}{a^2}\right) = \mu_1 e^{-\beta t}$$

$$u_0\left(\frac{u_0}{\left(a^2 - 1\right)^2} - \frac{r}{a^2}\right) = \mu_2 e^{-\beta t}$$



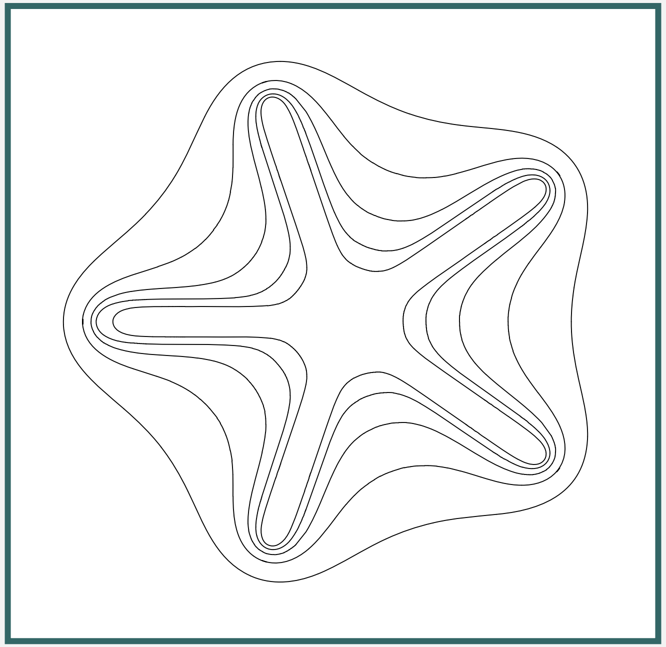
2^{nd} *example: Mineev's logarithmic maps*

$$z(w) = r(t)\omega + \sum_j \alpha_j \log\left(1 - \frac{w}{w_j}\right), \qquad \left|\omega_{\mathrm{j}}\right| > 1, \;\; \sum_j \alpha_j = 0$$

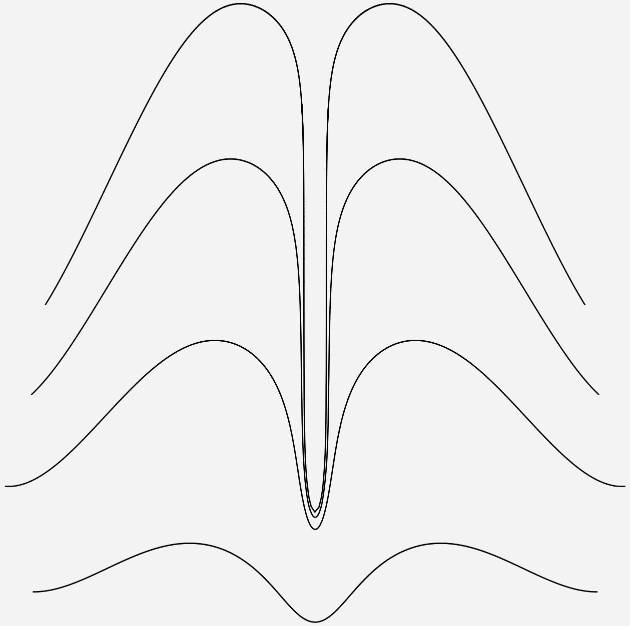
$$z\left(\frac{1}{w_j}\right) = const$$

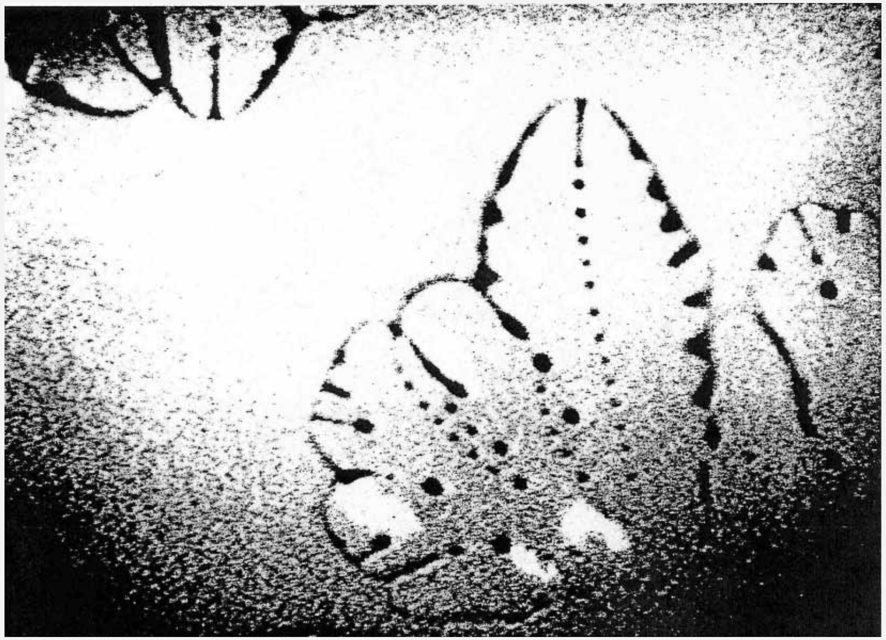
$$\alpha_j(t) = \alpha_j(0)\, e^{-\beta t}$$

$$\left|r(t)\right|^2 - r(t)\sum_j \frac{\bar{\alpha}_j(t)}{w_j} = t_0 e^{-\beta t}$$

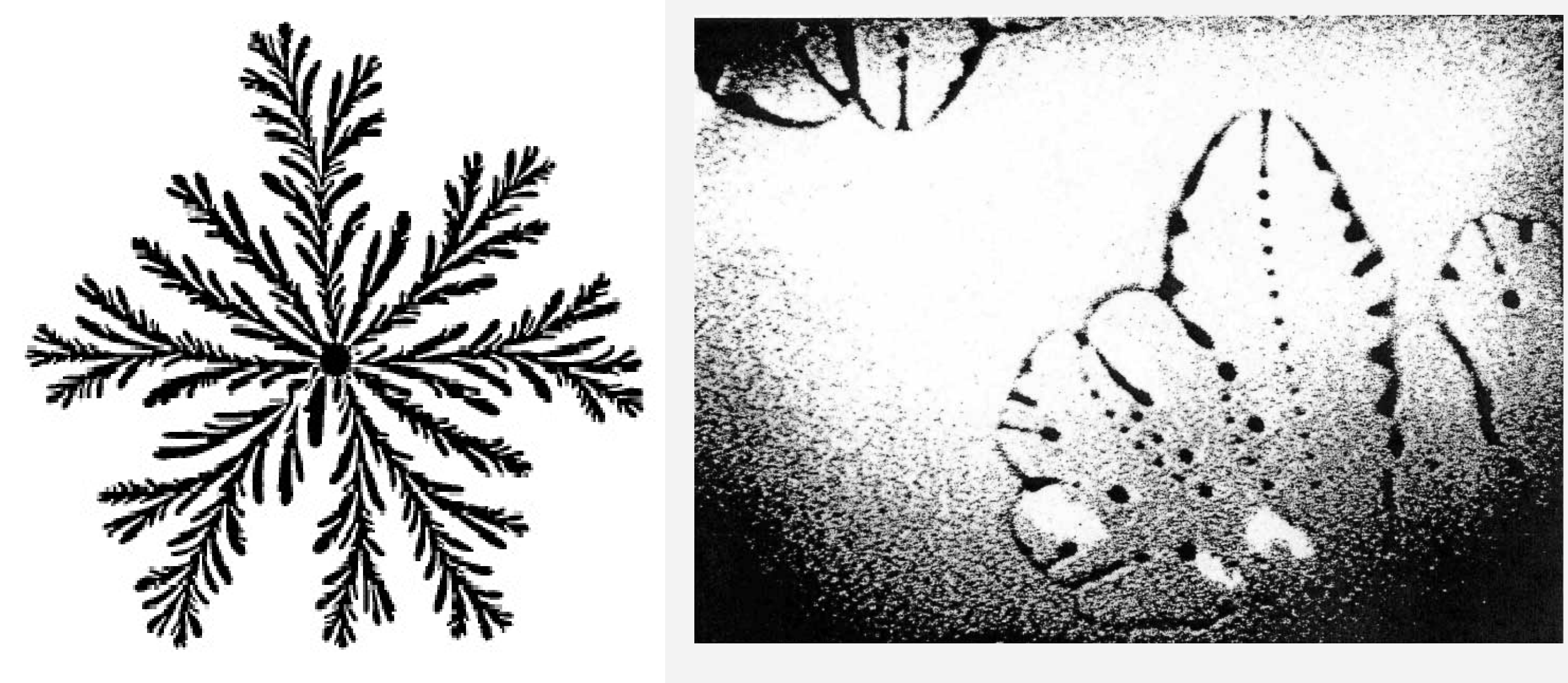


The "doublon"

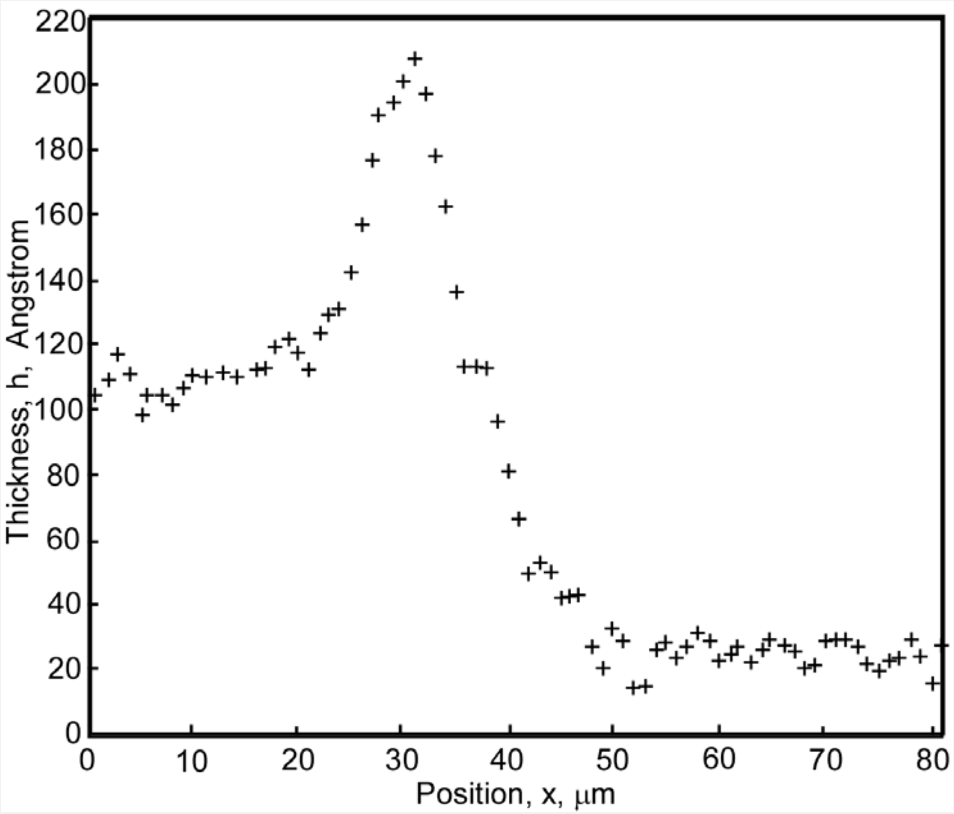


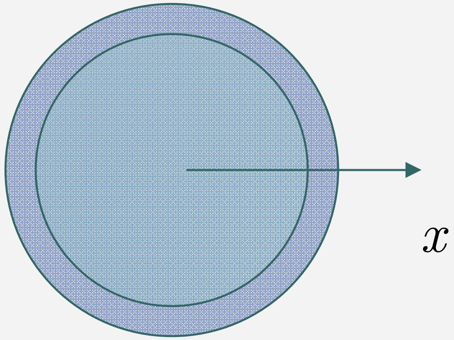


The selection problem



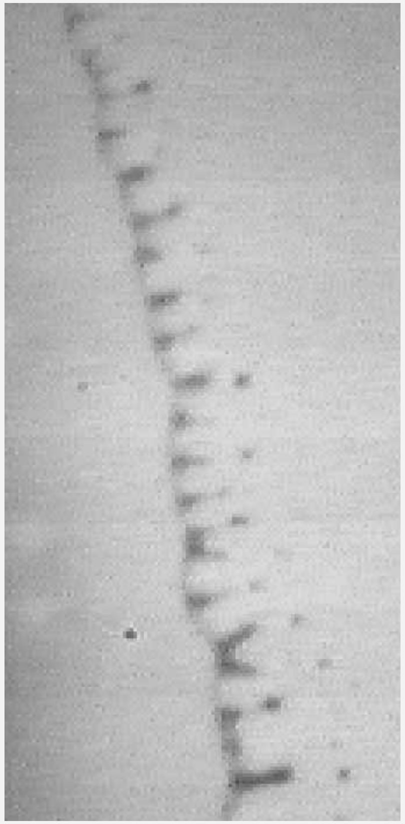
The rim instability

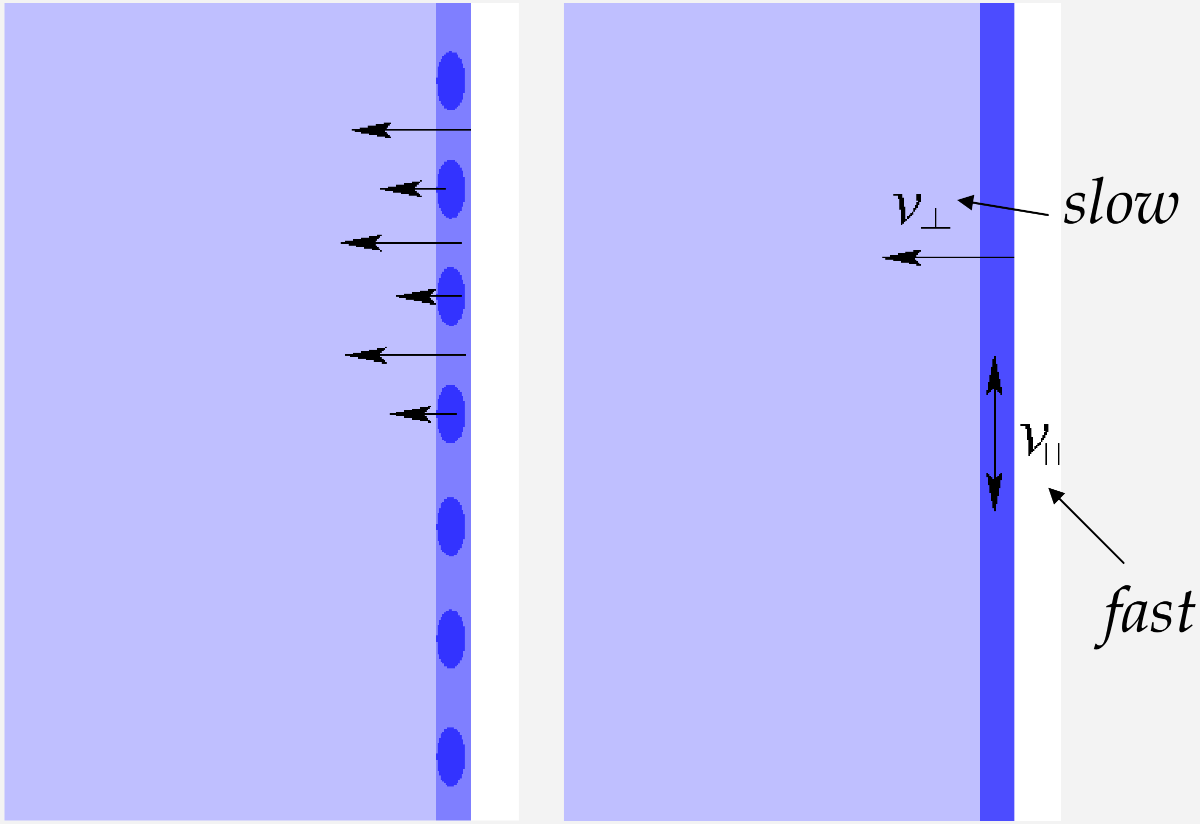




Rayleigh instability at the droplet's rim

(Leizerson, Lipson & Lyushnin, 2004)





Rayleigh instability (cont.)

Darcy's: $v_\parallel = -\alpha h^2 \nabla P$

Continuity: $\dfrac{dh}{dt} = -\nabla\left(h v_\parallel\right) = \alpha \nabla \cdot \left(h^3 \nabla P\right)$

Force balance $P = \dfrac{A}{h^3} + \gamma K$

van der Waals

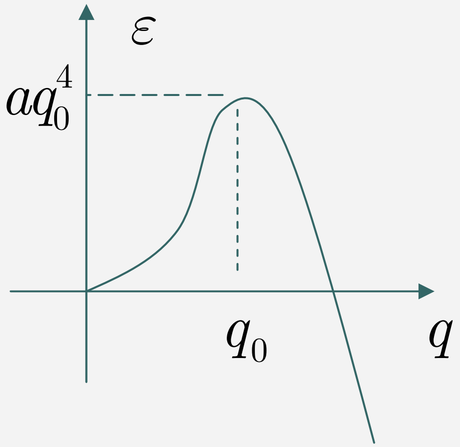
Surface tension

$K \sim -\nabla^2 h$

$$\frac{d\delta h}{dt} = -a\left(\nabla^4 + 2q_0^2\nabla^2\right)\delta h$$

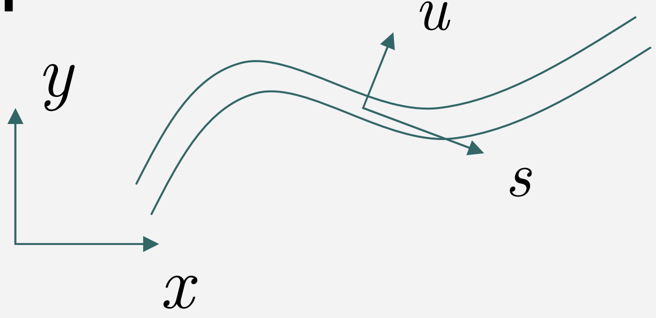
$\delta h \sim \delta h_0 e^{iqx+\varepsilon t}$ ⇨ $\varepsilon = a\left(2q_0^2 q^2 - q^4\right)$

$\varepsilon = \varepsilon_{\max}$ *for* $q = q_0$



Rayleigh instability on a curved rim

(Waveguides: Exner &Seba 89)



$$x = a(s) - ub'(s)$$

$$y = b(s) + ua'(s)$$

$$\left(a'(s)\right)^2 + \left(b'(s)\right)^2 = 1$$

$$\kappa(s) = b'(s)\,a''(s) - a'(s)\,b''(s)$$

$$\nabla^2 \to g^{-1}\frac{\partial}{\partial s}g^{-1}\frac{\partial}{\partial s} + g^{-1}\frac{\partial}{\partial u}g\frac{\partial}{\partial u} \qquad g = 1 + u\kappa(s)$$

Boundary conditions:

$$\delta h\big|_{u=-\frac{w}{2}} = 0, \qquad \nabla^2\delta h\big|_{u=-\frac{w}{2}} = 0$$

$$\left.\frac{\partial\delta h}{\partial u}\right|_{u=\frac{w}{2}} = 0, \qquad \left.\frac{\partial}{\partial u}\nabla^2\delta h\right|_{u=\frac{w}{2}} = 0$$

$H = -a\left(\nabla^4 + q_0^2\nabla^2\right)$ *is hermitian*

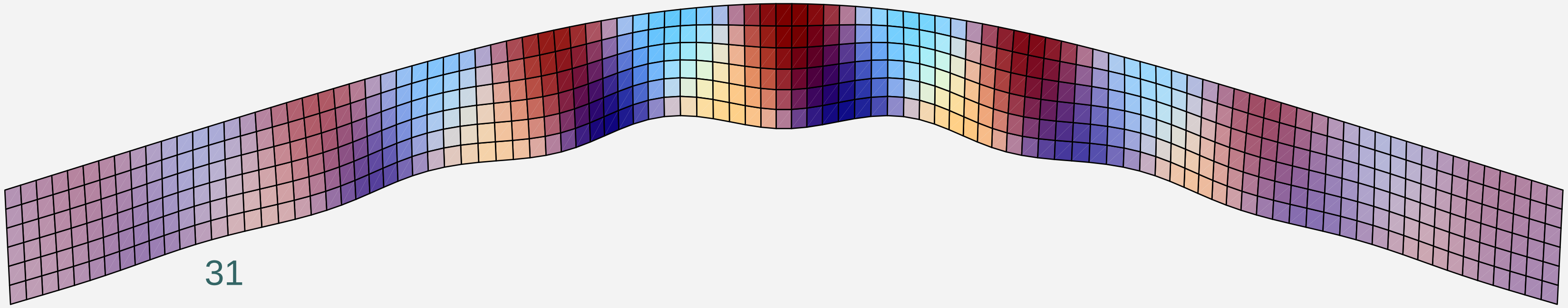
Variational approach

$$F[\psi] = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}$$

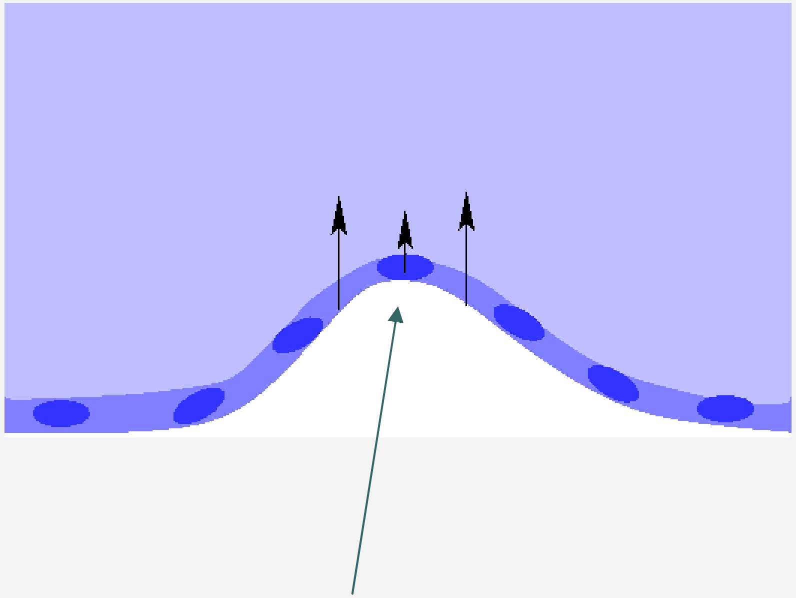
$$\kappa(s) = \kappa_0 e^{-\frac{s^2}{\sigma^2}}$$

$$\psi(s,u) = \left\{\cos\left(\frac{\pi}{4}\left(1-\frac{2u}{w}\right)\right) - \frac{w\kappa(s)}{4\pi^2}\left[\left(\pi^2\left(1+\frac{2u}{w}\right)\right)\cos\left(\frac{\pi}{4}\left(1-\frac{2u}{w}\right)\right) + 2\pi\left(1+\frac{2u}{w}\right)\sin\left(\frac{\pi}{4}\left(1-\frac{2u}{w}\right)\right)\right]\right\} f(s)$$

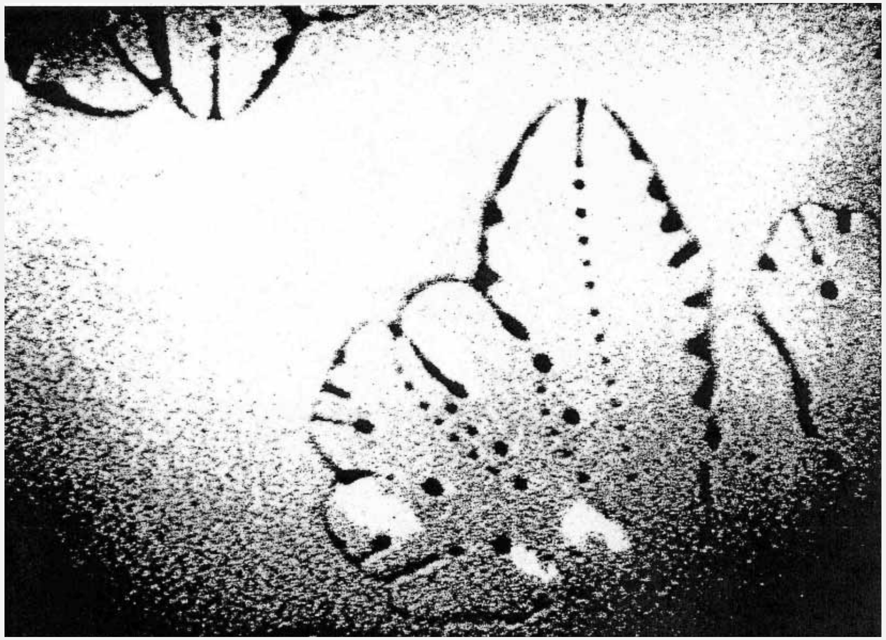
$$f(s) = \exp\left(-\frac{s^2}{a^2}\right) \times \left\{\begin{matrix}\cos(q_s s)\\ \sin(q_s s)\end{matrix}\right]$$

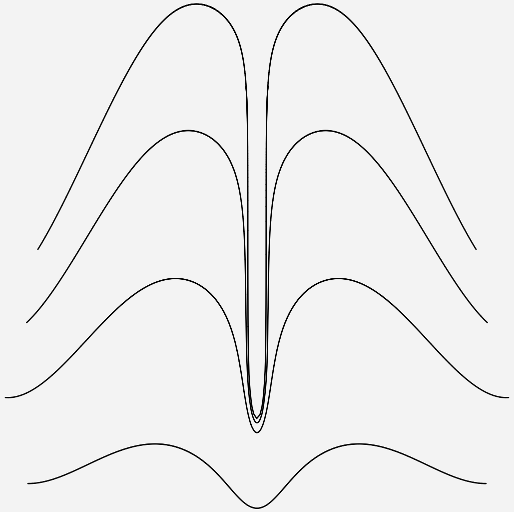


The “doublon”



Maximum curvature is a “nucleation center” which hinders cusps





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