Random Matrix Theory and the Askey Table↔Painlevé-Sakai Scheme

N.S. Witte

Department of Mathematics and Statistics, University of Melbourne

Outline

- averages in random matrix theory, determinantal point processes, mathematical physics, ...
- identification with integrable equations,
- characterisations of the solutions by ODE's, recurrences, ...
- solutions as non-linear generalisations of hypergeometric and basic hypergeometric functions,
- examples of classical and separatrix solutions to Painlevé systems,
- natural probabilistic setting of orthogonal, bi-orthogonal polynomial systems, ...
- regular semi-classical weights and the monodromy preserving property,
- the Askey Table,
- special non-uniform lattices and divided difference calculus,
- analogs of Painlevé equations

This research has been supported by the Australian Research Council.

Unitary Group Averages

Formal weight $w(z)$ and its Fourier coefficients $\{w_k\}_{k\in\mathbb{Z}}$

$$w(z) = \sum_{k=-\infty}^{\infty} w_k z^k, \quad w_k = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\zeta^{-k},$$

Average over the unitary group $U(n)$ with respect to the Haar measure

$$\left\langle \prod_{l=1}^{n} w(z_l) \right\rangle_{U(n)} = \frac{1}{(2\pi)^n n!} \int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_n \prod_{l=1}^{n} w(e^{i\theta_l}) \prod_{1\leq j<k\leq n} |e^{i\theta_j} - e^{i\theta_k}|^2$$
$$= \det[w_{i-j}]_{i,j=0,\dots,n-1} =: I_n[w]$$

First non-trivial case - $M = 3$

$$w(z) = z^{-i\omega_2}|1+z|^{2\omega_1}|1+tz|^{2\mu} \begin{cases} 1 & \theta \in (-\pi, \pi-\phi] \\ 1-\xi & \theta \in (\pi-\phi, \pi] \end{cases}$$

deformation variable, $t = e^{i\phi}$, complex parameters, $\mu, \omega_1, \omega_2, \xi$
$I_n[w]$ is a classical τ-function of the sixth Painlevé system [Forrester+W 2004]

Examples from Physics

μ	ω_1	ω_2	ξ	t	Interpretation
0	0	0	ξ	$e^{i\phi}$	Probability of exactly k eigenvalues of a random $n \times n$ unitary matrix with their phases in the sector $(\pi - \phi, \pi]$
$\frac{\sigma}{2}$	$\frac{\sigma}{2}$	0	0	$t = \lvert z \rvert^2$	2σ-th Moment of the characteristic polynomial $\lvert \det(U + z) \rvert^{2\sigma}$ of a random $n \times n$ unitary matrix
$\frac{1}{4}$	$\frac{1}{4}$	$\pm\frac{i}{2}$	0	$t^{\mp 1} = \sinh^2(2K_1)\sinh^2(2K_2)$	Diagonal spin-spin correlation $\langle \sigma_{0,0}\sigma_{n,n} \rangle$ of anisotropic square lattice Ising model
$\frac{1}{2}$	0	0	2	$e^{i\phi}$	Density matrix $\rho_{n+1}(\phi)$ of the homogeneous impenetrable Bose gas under periodic boundary conditions

Spectral Averages for $\beta = 2$ rank N Ensembles

Absolutely continuous weight $w(\lambda)$ with support I
Generalised Spectral Average

$$\tilde{E}_N(J;\mu;\xi) = \frac{1}{C_N} \prod_{j=1}^{N} \left(\int_I - \xi \int_J \right) d\lambda_j w(\lambda_j)(s-\lambda_j)^{\mu} \prod_{1 \le j < k \le N} (\lambda_j - \lambda_k)^2$$

$s \in \partial J, \ J \subset I$
Interpretations

$$\frac{(-1)^n}{n!} \frac{\partial^n}{\partial \xi^n} \tilde{E}_N(J;\mu=0;\xi)\Big|_{\xi=1} = \mathbb{P}(\sharp\{\lambda_j \in J\} = n)$$

$$\tilde{E}_N(J;\ldots;\mu=0;\xi) = \det(\mathbb{I} - \xi\, \mathbb{K}|_J)$$

$$\tilde{E}_N(J;\mu;\xi)\Big|_{\xi=0} = \mathbb{E}([\det(s\mathbb{I} - X)]^{\mu})$$

$\mathbb{K}$ integral operator with kernel

$$K(x,y) = \frac{a_{N-1}}{a_N} [w(x)w(y)]^{1/2} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y}$$

orthogonal polynomial system $\{p_n(x)\}_{n=0}^{\infty}$ wrt $w(x)$
E.g. Gaussian Unitary Ensemble (GUE) corresponds to

$$w(\lambda) = e^{-\lambda^2}, \quad I = (-\infty, \infty), \quad J = (s, \infty)$$

$\tilde{E}_N(J;\mu;\xi)$ is a classical $\mathrm{P_{IV}}$ τ-function [Forrester+W. 2001]

Nonlinear Differential Equations

Application of random matrix theory to the study of moments of the derivative of the Riemann zeta-function [Conrey, Rubinstein and Snaith 2005] utilising the association with the derivatives of characteristic polynomials for random unitary matrices. With U a Haar distributed element of the unitary group $U(N)$, and $e^{i\theta_1}, \dots, e^{i\theta_N}$ its eigenvalues, let

$$\Lambda_A(s) = \prod_{j=1}^{N} (1 - se^{-i\theta_j})$$

Asymptotic expressions

$$\langle |\Lambda_A'(1)|^{2k} \rangle_{A \in U(N)} \underset{N \to \infty}{\sim} b_k N^{k^2+2k}$$

Identification with a particular hard edge gap probability

$$b_k = \frac{(-1)^k}{k! \prod_{j=1}^{k} \frac{(k+j-1)!}{j!}} \sum_{h=0}^{k} \binom{k}{h} (k+h)! [x^{k+h}] \tilde{E}_2^{\rm hard}(4x; k, k; \xi = 1)$$

with

$$\tilde{E}_2^{\rm hard}(4x; k, k; \xi = 1) = \exp\left(- \int_0^{4x} \frac{ds}{s} \, (\sigma_{\rm III'}(s) + k^2) \right)$$

where $\sigma_{\rm III'}(s)$ satisfies the particular σ-Painlevé $\rm III'$ equation

$$(s\sigma_{\rm III'}'')^2 + \sigma_{\rm III'}'(4\sigma_{\rm III'}' - 1)(\sigma_{\rm III'} - s\sigma_{\rm III'}') - \frac{k^2}{16} = 0$$

subject to the boundary condition

$$\sigma_{\rm III'}(s) \underset{s \to 0}{\sim} -k^2 + \frac{s}{8} + {\rm O}(s^2), \quad k \in \mathbb{N}$$

Recurrences in Matrix Rank

Average of the 2μ-th power, or equivalently the 2μ-th moment, of the absolute value of the characteristic polynomial for the CUE $U \in U(N)$ of rank N

$$F_N^{\mathrm{CUE}}(u;\mu) := \left\langle \prod_{l=1}^{N} |u+z_l|^{2\mu} \right\rangle_{\mathrm{CUE}_N}$$

General moments are given by the system of recurrences

$$\frac{F_{N+1}^{\mathrm{CUE}} F_{N-1}^{\mathrm{CUE}}}{(F_N^{\mathrm{CUE}})^2} = 1 - |u|^{2N} r_N^2$$

with initial values

$$F_0^{\mathrm{CUE}} = 1, \qquad F_1^{\mathrm{CUE}} = {}_2F_1(-\mu,-\mu;1;|u|^2)$$

and

$$\begin{aligned} 2|u|^{2N} r_N r_{N-1} - |u|^2 - 1 \\ = \frac{1-|u|^{2N} r_N^2}{r_N} \left[(N+1+\mu)|u|^2 r_{N+1} + (N-1+\mu) r_{N-1}\right] \\ - \frac{1-|u|^{2(N-1)} r_{N-1}^2}{r_{N-1}} \left[(N+\mu)|u|^2 r_N + (N-2+\mu) r_{N-2}\right] \end{aligned}$$

subject to the initial values

$$r_0 = 1, \quad r_1 = -\mu \frac{{}_2F_1(-\mu,-\mu+1;2;|u|^2)}{{}_2F_1(-\mu,-\mu;1;|u|^2)}$$

Identities

Average of the powers of the characteristic polynomial

$$F_N(s;\mu) := \left\langle \prod_{l=1}^{N} (s-\lambda_l)^{\mu} \right\rangle_{GUE} = \tilde{E}_N(s;\mu;\xi=0)$$

Obey the duality relation [Forrester+W 2001]

$$\frac{F_N(s;\mu)}{F_N(s_0;\mu)} = \frac{F_\mu(is;N)}{F_\mu(is_0;N)}$$

for all $\mu, N \in \mathbb{N}$. Identity implies the integral identity

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_N \prod_{j=1}^{N} e^{-x_j^2}(s-x_j)^{\mu} \prod_{1\le j<k\le N} (x_k-x_j)^2$$

$$= C \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_\mu \prod_{j=1}^{a} e^{-x_j^2}(s-ix_j)^{N} \prod_{1\le j<k\le a} (x_k-x_j)^2$$

and the determinant identity

$$\det\left[\int_{-\infty}^{\infty} (s-x)^{\mu} x^{j+k} e^{-x^2}\,dx\right]_{j,k=0,\dots,N-1} = C \det\left[\int_{-\infty}^{\infty} (s-ix)^{N} x^{j+k} e^{-x^2}\,dx\right]_{j,k=0,\dots,\mu-1}$$

Classical Solutions

Expansion of the $\mathrm{P_{VI}}$ τ-function as $t \to 0$ in the domain $\{t \in \mathbb{C} | 0 < |t| < \varepsilon, |\arg(t)| < \phi\}$ for all $\varepsilon > 0$ and any $\phi > 0$ [Jimbo 1982]

$$
\begin{aligned}
\tau(t) \sim C t^{(\sigma_{0t}^2 - \theta_0^2 - \theta_t^2)/4} \\
\times \Bigg\{ 1 + \frac{(\theta_0^2 - \theta_t^2 - \sigma_{0t}^2)(\theta_\infty^2 - \theta_1^2 - \sigma_{0t}^2)}{8\sigma_{0t}^2} t \\
- \hat{s} \frac{[\theta_0^2 - (\theta_t - \sigma_{0t})^2][\theta_\infty^2 - (\theta_1 - \sigma_{0t})^2]}{16\sigma_{0t}^2 (1+\sigma_{0t})^2} t^{1+\sigma_{0t}} \\
- \hat{s}^{-1} \frac{[\theta_0^2 - (\theta_t + \sigma_{0t})^2][\theta_\infty^2 - (\theta_1 + \sigma_{0t})^2]}{16\sigma_{0t}^2 (1-\sigma_{0t})^2} t^{1-\sigma_{0t}} + \mathrm{O}(|t|^{2(1-\Re(\sigma_{0t}))}) \Bigg\}
\end{aligned}
$$

For $I_N[w]$ the monodromy invariants for either case are $(\omega, \bar{\omega} := \omega_1 \pm i\omega_2)$

$$
\sigma_{0t} = N - \mu + \bar{\omega}, \quad \sigma_{t1} = 2\mu + 2\omega_1, \quad \sigma_{01} = N - \mu + \omega
$$

$$
\begin{aligned}
&\mathrm{Case}(A): \llcorner \quad \theta_0 = -\mu - \omega, \quad \theta_t = N + 2\omega_1, \quad \theta_1 = N + 2\mu, \quad \theta_\infty = -\mu - \bar{\omega}, \\
&\mathrm{Case}(B): M_t = (-)^N I \quad \theta_0 = \mu - \bar{\omega}, \quad \theta_t = N, \quad \theta_1 = N + 2\mu + 2\omega_1, \quad \theta_\infty = \mu - \omega, \\
&\mathrm{Case}(C): \urcorner \quad \theta_0 = -2\omega_1, \quad \theta_t = N + \mu + \omega, \quad \theta_1 = N + \mu + \bar{\omega}, \quad \theta_\infty = 2\mu
\end{aligned}
$$

Separatrix Solutions

Global asymptotics of $\mathrm{P_{II}}$ [Kapaev 1992], [Kitaev 1994], [Its+Kapaev 2000], [Its+Kapaev 2001]
Monodromy data $\{s_1, s_2, s_3\}$ satisfying Monodromy Manifold

$$\mathfrak{M}_{\mathrm{II}}(s_1, s_2, s_3) = s_1 - s_2 + s_3 + s_1 s_2 s_3 + 2\sin\pi\alpha = 0$$

Special $\mathrm{P_{II}}$ Solutions

- Rational solutions $s_1 = s_2 = s_3 = 0,\ \alpha \in \mathbb{Z}$ Yablonskii-Voro'bev polynomials
- Classical solutions $s_1 = -s_2 = s_3 = (-1)^{n+1},\ \alpha = n + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ Determinants constructed with Airy function entries
- Separatrix solutions $s_2 = 0,\ s_1 = 1/s_3 = e^{i\pi\epsilon(\alpha+\frac{1}{2})}\ \forall\alpha,\ \epsilon = \pm 1$

Soft-edge Regime

Scaling from GUE $\rightarrow$ Soft Edge $\leftrightarrow$ degeneration $\mathrm{P_{IV}}$ to $\mathrm{P_{II}}$

$$\tilde{E}^{\rm soft}(s;\mu;\xi) = \lim_{N\to\infty} e^{-\mu t^2}\tilde{E}_N(t;\mu;\xi)\Big|_{t=\sqrt{2N}+s/\sqrt{2}N^{1/6}}$$

$\xi = 1$, $\tilde{E}^{\rm soft}(s;\mu)$; $\xi = 0$, $F^{\rm soft}(s;\mu)$ are $\mathrm{P_{II}}$ τ-functions with $\alpha = \mu - \frac{1}{2}$ [Forrester+W. 2001]

Two *separatrix/truncated* solutions [Kapaev 1992], [Kitaev 1994]

$$\sigma(t;\mu) \underset{t\to-\infty}{\sim} \frac{1}{4}t^2 + \frac{4\mu^2-1}{8t} + \ldots - \epsilon 2^{-\frac{3}{2}}\sin(\pi\mu)\frac{e^{-\frac{1}{3}(-2t)^{3/2}}}{(-2t)^{-\frac{1}{4}}}\left(1+\mathrm{O}((-t)^{-3/2})\right)$$

$$= u(t;\mu) \quad \text{with} \quad \epsilon = 1$$

i.e. symmetry broken and exponentially small terms present

$$\sigma(t;\mu) \underset{t\to\infty}{\sim} \epsilon\mu t^{1/2} - \frac{\mu^2}{4t} + \ldots$$

$$+(\epsilon+1)2^{3\epsilon(\mu-\frac{1}{2})-\frac{5}{2}}\frac{\cos(\pi\mu)}{\pi}\Gamma(\frac{\epsilon+1}{2}-\epsilon\mu)\frac{e^{-\frac{4}{3}t^{3/2}}}{t^{-\frac{1}{4}(3\epsilon+1)+\frac{3}{2}\epsilon\mu}}\left(1+\mathrm{O}(t^{-3/2})\right)$$

$$= v(t;\mu) \quad \text{with} \quad \epsilon = -1$$

i.e. no exponentially small terms exist

Conjecture

All the generalised spectral averages of the scaled Random Matrix ensembles (Soft Edge, Hard Edge, Bulk) are transcendental, separatrix solutions of the Painlevé II, III and V systems.

Orthogonal Systems

Orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ with respect to the weight $w(x)$ on support C

$$\int_C dx\; w(x)p_m(x)p_n(x) = \delta_{m,n}$$

Stieltjes Function

$$f(x) := \int_C dz \frac{w(z)}{x-z} = \sum_{n=0}^{\infty} \frac{\mu_n}{x^{n+1}}$$

Associated Function

$$q_n(x) := \int_C dz \frac{w(z)}{x-z} p_n(z)$$

Matrix [Its, Fokas, Kitaev 1991]

$$Y_n(x;t) := \begin{pmatrix} p_n(x) & q_n(x)/w(x) \\ p_{n-1}(x) & q_{n-1}(x)/w(x) \end{pmatrix},$$

[Hamburger, Favard] *The orthogonal system* $\{Y_n\}_{n=0}^{\infty}$ *exists if and only if* $H_n := \det(\mu_{j+k})_{0\le j,k\le n-1} \neq 0$ *for all* $n \in \mathbb{N}$.

Determinantal point processes

Definition: *A simple point process such that it's joint intensities - correlation functions - satisfy*

$$\rho_n(x_1, \ldots, x_n) = \det(K(x_j, x_k))_{1 \leq j,k \leq n}$$

for every $n \geq 1$ *and* $x_1, \ldots, x_n \in \Lambda$.

Lemma: *Let* $\{p_k\}_{k=1}^{n}$ *be an orthonormal set in* $L^2(\Lambda)$. *Then there exists a determinantal process with kernel* $K(x,y) = \sum_{k=1}^{n} p_k(x) p_k(y)$.

See Ben Hough, Krishnapur and Peres, Probability Surveys, **3** (2006) 206-229

Structural Relations for Orthogonal Systems

There-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_n(x)$$

or matrix Y_n satisfies the recurrence relation in n

$$Y_{n+1} := K_n Y_n = \frac{1}{a_{n+1}} \begin{pmatrix} x - b_n & -a_n \\ a_{n+1} & 0 \end{pmatrix} Y_n$$

The Casoratians of the solutions p_n, q_n are

$$p_n(z)q_{n-1}(z) - p_{n-1}(z)q_n(z) = \frac{1}{a_n}$$

[Bonan & Clark 1990, Bauldry 1990]*Assume that the weight satisfies the moment conditions*

$$\int_C dx\ w(x) \frac{\frac{d}{dz}\log w(z) - \frac{d}{dx}\log w(x)}{z-x} x^j \neq \infty, \quad j \in \mathbb{N}$$

Then the matrix Y_n *satisfies the differential relation with respect to the spectral variable* x

$$\frac{d}{dx} Y_n = A_n Y_n$$

Regular Semi-classical Class of Weights

Definition - regular semi-classical class - equivalent to Pearson equation

$$\frac{1}{w(x)}\frac{d}{dx}w(x) = \frac{2V(x)}{W(x)} = \sum_{j=1}^{M}\frac{\rho_j}{x-x_j}, \quad x_j, \rho_j \in \mathbb{C}$$

$V(x)$, $W(x)$ are polynomials

(i) $\deg(W) \geq 2$,

(ii) $\deg(V) < \deg(W) = M$,

(iii) the M zeros of $W(x)$, $\{x_1, \ldots, x_M\}$ are distinct, and

(iv) the residues $\rho_j = 2V(x_j)/W'(x_j) \notin \mathbb{Z}_{\geq 0}$

Semi-classical class follows from degenerations/limits of a regular S-C weight, e.g. $M = 2$ case

OPS	$w(x)$	$W(x)$	$V(x)$
Jacobi	$(1-x)^{\alpha}(1+x)^{\beta}$	x^2-1	$\frac{1}{2}(\alpha-\beta)+\frac{1}{2}(\alpha+\beta)x$
Laguerre	$x^{\alpha}e^{-x}$	x	$\frac{1}{2}(\alpha-x)$
Hermite	e^{-x^2}	1	$-x$

Regular Semi-classical Class for Orthogonal systems

Parameterisation of spectral matrix

$$A_n = \frac{1}{W(x)} \begin{pmatrix} \Omega_n(x) - V(x) & -a_n\Theta_n(x) \\ a_n\Theta_{n-1}(x) & -\Omega_n(x) - V(x) \end{pmatrix}$$

$\Theta_n(x)$ and $\Omega_n(x)$ are polynomials of degree $M-1, M$ respectively, independent of n, with

$$\Theta_n(x) = (2n+1+m_0)x^{M-1} + \dots$$

$$\Omega_n(x) = (n + \frac{1}{2}m_0)x^M + \dots$$

where $m_0 = \sum_1^M \rho_j$.
Therefore

$$A_n = \sum_{j=1}^{M} \frac{A_n(x_j)}{x - x_j}$$

where

$$A_{n,j} := A_n(x_j) = a_n\rho_j \begin{pmatrix} p_{n-1}(x_j)q_n(x_j) & -p_n(x_j)q_n(x_j) \\ p_{n-1}(x_j)q_{n-1}(x_j) & -p_n(x_j)q_{n-1}(x_j) \end{pmatrix}$$

Compatibility Relations for Orthogonal Systems

Compatibility of spectral and recurrence structures implies

$$K_n' = A_{n+1}K_n - K_nA_n$$

[Magnus 1995] *The coefficients* $\Theta_n(x), \Omega_n(x)$ *satisfy the recurrence relations*

$$\Omega_{n+1}(x) + \Omega_n(x) = (x - b_n)\Theta_n(x)$$

and

$$[\Omega_{n+1}(x) - \Omega_n(x)](x - b_n) = W(x) + a_{n+1}^2\Theta_{n+1}(x) - a_n^2\Theta_{n-1}(x)$$

[W 2004] *Evaluated at a finite singular point,* x_j*, the coefficients* $\Theta_n(x)$*,* $\Omega_n(x)$ *satisfy the bilinear relations*

$$\Omega_n^2(x_j) - V^2(x_j) = a_n^2\Theta_n(x_j)\Theta_{n-1}(x_j)$$

Deformation Structures for Orthogonal Systems

Parameterise the free singularities $x_j(t)$ as arbitrary trajectories

$$\frac{d}{dt} = \sum_{j=1}^{M} \dot{x}_j \frac{\partial}{\partial x_j} \quad \dot{x}_j := \frac{dx_j}{dt}$$

[Magnus 1995] The deformation derivatives of the system with respect to arbitrary deformations of the singularities x_j are given by

$$\frac{d}{dt} Y_n = B_n Y_n = \left[B_\infty - \sum_{j=1}^{M} \dot{x}_j \frac{A_{n,j}}{x - x_j} \right] Y_n$$

where

$$B_\infty = \begin{pmatrix} \frac{\dot{\gamma}_n}{\gamma_n} & 0 \\ 0 & -\frac{\dot{\gamma}_{n-1}}{\gamma_{n-1}} \end{pmatrix}$$

Compatibility of spectral and deformation structures

The residue matrices satisfy a system of integrable, non-linear partial differential equations, the Schlesinger equations,

$$\dot{A}_{n,j} = \left[B_\infty, A_{n,j}\right] + \sum_{\substack{k \neq j \\ 1 \leq k \leq M}} \frac{\dot{x}_j - \dot{x}_k}{x_j - x_k} \left[A_{n,k}, A_{n,j}\right], \qquad j = 1, \ldots, M$$

$$\dot{A}_{n,\infty} = [B_\infty, A_{n,\infty}]$$

Isomonodromy for Garnier Systems

Monodromy Matrix

$$Y_n|_{x_j+\delta e^{2\pi i}} = Y_n|_{x_j+\delta} M_j$$

Upper triangular form

$$M_j = \begin{pmatrix} 1 & c_j(1 - e^{-2\pi i \rho_j}) \\ 0 & e^{-2\pi i \rho_j} \end{pmatrix}$$

Monodromy preserving deformations c_j is independent of x_k

Christoffel-Uvarov Transformations for Orthogonal Systems

General rational modification of the weight

$$w\left[\begin{array}{ccc} \alpha_1 & \dots & \alpha_K \\ \beta_1 & \dots & \beta_L \end{array};x\right] = \frac{\prod_{i=1}^{K}(x-\alpha_i)}{\prod_{k=1}^{L}(x-\beta_k)} w(x)$$

Matrix Generators $R_n^+(x;\alpha), R_n^-(x;\beta), \dots$ for $K=1, L=0$ and $K=0, L=1$, ...

$$Y_n\left[\begin{array}{c} 1,\cdot \\ 0,\cdot \end{array};x\right] = R_n^+(x;\alpha)Y_n(x), \qquad Y_n\left[\begin{array}{c} 0,\cdot \\ 1,\cdot \end{array};x\right] = R_n^-(x;\beta)Y_n(x)$$

$R-$matrices

$$R_n^+(x;\alpha) = \begin{pmatrix} \frac{\hat{A}_n}{a_n}p_n(\alpha) & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{x-\alpha}\begin{pmatrix} \frac{a_n}{a_{n+1}}\hat{A}_n p_{n-1}(\alpha) & -\frac{a_n}{a_{n+1}}\hat{A}_n p_n(\alpha) \\ \hat{A}_{n-1}p_{n-1}(\alpha) & -\hat{A}_n p_n(\alpha) \end{pmatrix}$$

where $\hat{A}_n^2 = -\gamma_n/\gamma_{n+1}p_n(\alpha)p_{n+1}(\alpha)$

$$R_n^-(x;\beta) = \begin{pmatrix} \check{A}_n q_{n-1}(\beta) & -\check{A}_n q_n(\beta) \\ \frac{a_n}{a_{n-1}}\check{A}_{n-1}q_{n-1}(\beta) & -\frac{a_n}{a_{n-1}}\check{A}_{n-1}q_n(\beta) \end{pmatrix} + (x-\beta)\begin{pmatrix} 0 & 0 \\ 0 & -\frac{\check{A}_{n-1}}{a_{n-1}}q_{n-1}(\beta) \end{pmatrix}$$

where $\check{A}_n^2 = -\gamma_{n-1}/\gamma_n q_{n-1}(\beta)q_n(\beta)$

Schlesinger Transformations for Orthogonal Systems

C-U transformations with $\alpha, \beta, \rightarrow x_j$
Three-term recurrence and ρ_j exponent transformations in terms of monodromy exponents

$n \mapsto n \pm 1$	$\theta_0 \mapsto \theta_0 \pm 1$, $\theta_\infty \mapsto \theta_\infty \pm 1$
$\rho_j \mapsto \rho_j \pm 1$	$\theta_j \mapsto \theta_j \mp 1$, $\theta_\infty \mapsto \theta_\infty \pm 1$
$n \mapsto n \pm 1$, $\rho_j \mapsto \rho_j \mp 1$	$\theta_0 \mapsto \theta_0 \pm 1$, $\theta_j \mapsto \theta_j \pm 1$

Compatibility of recurrence matrix and the Schlesinger transformation

$$R^{j\pm}_{n+1}(x;\rho_j)K_n(x;\rho_j) = K_n(x;\rho_j \pm 1)R^{j\pm}_n(x;\rho_j)$$

Compatibility of spectral matrix and the Schlesinger transformation

$$\frac{d}{dx}R^{j\pm}_n(x;\rho_j) + R^{j\pm}_n(x;\rho_j)A_n(x;\rho_j) = A_n(x;\rho_j \pm 1)R^{j\pm}_n(x;\rho_j)$$

Compatibility of deformation matrix and the Schlesinger transformation

$$\frac{d}{dt}R^{j\pm}_n(x;\rho_j) + R^{j\pm}_n(x;\rho_j)B_n(x;\rho_j) = B_n(x;\rho_j \pm 1)R^{j\pm}_n(x;\rho_j)$$

The Askey Table

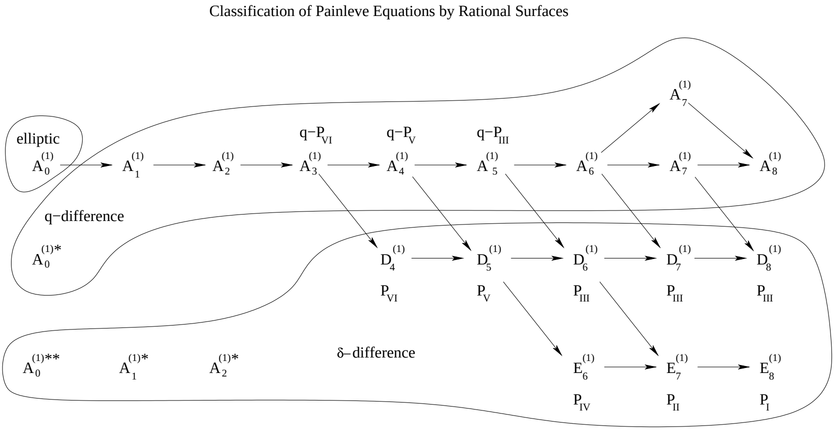
HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

${}_4F_3$	Wilson		Racah		
${}_3F_2$	Continuous Dual Hahn	Continuous Hahn	Hahn	Dual Hahn	
${}_2F_1$	Meixner − Pollaczek	Jacobi	Meixner	Krawtchouk	
${}_1F_1$ / ${}_2F_0$	Laguerre		Charlier		
${}_2F_0$	Hermite				

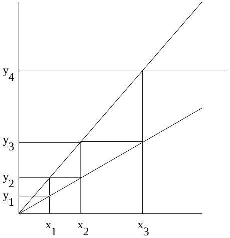
BASIC HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

${}_4\phi_3$	Askey−Wilson	q−Racah								
${}_4\phi_3$ / ${}_3\phi_2$	Continuous Dual q−Hahn	Continuous q−Hahn	Big q−Jacobi	q−Hahn	Dual q−Hahn					
${}_4\phi_3$ / ${}_2\phi_1$	Al−Salam Chihara	q−Meixner Pollaczek	Continuous q−Jacobi	Big q−Laguerre	Little q−Jacobi	q−Meixner	Quantum q−Krawtchouk	q−Krawtchouk	Affine q−Krawtchouk	Dual q−Krawtchouk
${}_2\phi_1$ / ${}_2\phi_0$ / ${}_1\phi_1$	Continuous big q−Hermite	Continuous q−Laguerre	Little q−Laguerre	q−Laguerre	Alternative q−Charlier	q−Charlier	Al−Salam Carlitz I	Al−Salam Carlitz II		
${}_2\phi_0$ / ${}_1\phi_1$	Continuous q−Hermite	Stieltjes − Wigert	Discrete q−Hermite I	Discrete q−Hermite II						

The Sakai Scheme



q-linear lattice and divided difference calculus



Lattice [Nikiforov, Uvarov & Suslov 1991, Magnus 1988]

$$x_s = q^s, \quad s \in \mathbb{Z}, \quad |q| < 1$$

Divided difference operator and Mean operator

$$D_q f(x) := \frac{f(qx) - f(x)}{(q-1)x}, \quad M_q f(x) := \frac{1}{2}[f(qx) + f(x)]$$

Jackson q-sum

$$\int_0^a d_q x \, f(x) := a(1-q) \sum_{s=0}^{\infty} q^s f(aq^s)$$

q-Pochhammer symbol

$$(x;q)_n = \prod_{j=0}^{n-1} (1 - xq^j)$$

D_q-semi-classical weights

Definition: Let the D_q*-semi-classical weight* satisfy

$$WD_q w = 2VM_q w$$

or equivalently

$$\frac{w(qx)}{w(x)} = \frac{W + (q-1)xV}{W - (q-1)xV}$$

for $V(x), W(x)$ irreducible polynomials
E.g. little q-Jacobi weight

$$w(x) = x^{\alpha} \frac{(qx;q)_{\infty}}{(q^{\beta+1}x;q)_{\infty}}$$

Weight data $M = 2$

$$2W = x\left[q^{\alpha} + 1 - (q^{\alpha+\beta} + 1)qx\right], \quad 2(q-1)V = q^{\alpha} - 1 - (q^{\alpha+\beta} - 1)qx$$

Spectral Structure

The matrix Y_n satisfies the *spectral divided difference* equation [Magnus 1988, Chen & Ismail 1991, W 2009]

$$D_q Y_n(x) := A_n M_q Y_n(x) = \frac{1}{W_n(x)} \begin{pmatrix} \Omega_n(x) & -a_n\Theta_n(x) \\ a_n\Theta_{n-1}(x) & -\Omega_n(x) - 2V(x) \end{pmatrix} M_q Y_n(x)$$

Definition: regular D_q-semi-classical case W_n, Θ_n, Ω_n are polynomials in x with

$$\deg_x W_n = \deg_x W = M, \quad \deg_x \Omega_n = \deg_x V = M - 1, \quad \deg_x \Theta_n = M - 2$$

No meaning attached to the zeros of W_n as singular points

Compatibility and Bilinear relations

Compatibility implies

$$K_n(qx)\left(1-\tfrac{1}{2}(q-1)x\,A_n\right)^{-1}\left(1+\tfrac{1}{2}(q-1)x\,A_n\right)$$
$$=\left(1-\tfrac{1}{2}(q-1)x\,A_{n+1}\right)^{-1}\left(1+\tfrac{1}{2}(q-1)x\,A_{n+1}\right)K_n(x)$$

The spectral coefficients satisfy recurrence relations in n [Magnus 1988, W 2009]

$$W_{n+1}=W_n+\tfrac{1}{4}(q-1)^2x^2\Theta_n$$
$$\Omega_{n+1}+\Omega_n+2V=(M_qx-b_n)\Theta_n$$

$$(W_n\Omega_{n+1}-W_{n+1}\Omega_n)(M_qx-b_n)$$
$$=-\tfrac{1}{4}(q-1)^2x^2\Omega_{n+1}\Omega_n+W_nW_{n+1}+a_{n+1}^2W_n\Theta_{n+1}-a_n^2W_{n+1}\Theta_{n-1}$$

Bilinear relation

$$[2W_n-W+(q-1)x(\Omega_n+V)]\,[2W_n-W-(q-1)x(\Omega_n+V)]+a_n^2(q-1)^2x^2\Theta_n\Theta_{n-1}$$
$$=W^2-(q-1)^2x^2V^2$$

D_q-semi-classical Deformation Structure

The *deformed D_q-semi-classical weight* satisfies [W 2009]

$$RD_{q,t}w = 2SM_{q,t}w,$$

or equivalently

$$\frac{w(x;qt)}{w(x;t)} = \frac{R+(q-1)tS}{R-(q-1)tS}$$

for $S(x;t), R(x;t)$ irreducible in x and t.
E.g. deformed little q-Jacobi

$$w(x;t) = x^{\alpha}\frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\frac{(qx/t;q)_\infty}{(q^{\gamma+1}x/t;q)_\infty}$$

Weight data $M=3$

$$\begin{aligned}
2W &= q^{\alpha}(1-q^{\beta+1}x)(1-q^{\gamma+1}x/t)x+(1-qx)(1-qx/t)x,\\
2(q-1)V &= q^{\alpha}(1-q^{\beta+1}x)(1-q^{\gamma+1}x/t)-(1-qx)(1-qx/t)\\
2R &= 2x-(q^{-\gamma}+1)t\\
2(q-1)S &= q^{-\gamma}-1
\end{aligned}$$

Deformation Structure

OPS corresponding to a deformed D_q-semi-classical weight satisfies the *deformation divided difference* equation [W 2009]

$$D_{q,t} Y_n := B_n M_{q,t} Y_n = \frac{1}{R_n} \begin{pmatrix} \Gamma_n & \Phi_n \\ \Psi_n & \Xi_n \end{pmatrix} M_{q,t} Y_n$$

regular, deformed D_q-semi-classical case the deformation coefficients $R_n, \Gamma_n, \Phi_n, \Psi_n, \Xi_n$ are polynomials in the spectral variable x with degrees independent of n

$$\deg_x R_n = \deg_x \Gamma_n = \deg_x \Xi_n = \max(\deg_x R, \deg_x S),$$
$$\deg_x \Phi_n = \deg_x \Psi_n = \max(\deg_x R, \deg_x S) - 1$$

Deformation coefficients satisfy the linear identity and the trace identity

$$\Psi_n = -\frac{a_n}{a_{n-1}} \Phi_{n-1}$$
$$(q-1)t\,(\Gamma_n + \Xi_n) = 2H_n \left[\frac{R + (q-1)tS}{a_n(t)} - \frac{R - (q-1)tS}{a_n(qt)} \right]$$

E.g. deformed little q-Jacobi $L = \deg_x R = 1$ that the higher deformation coefficients can be parameterised thus

$$\mathfrak{R}_\pm = r_{1\pm} x + r_{0\pm}, \quad \mathfrak{P}_\pm = p_\pm$$

Further compatibility and Bilinear relations

The recurrence matrix and the deformation matrix satisfy [W 2009]

$$K_n(;qt)\left(1-\tfrac{1}{2}(q-1)tB_n\right)^{-1}\left(1+\tfrac{1}{2}(q-1)tB_n\right) = \left(1-\tfrac{1}{2}(q-1)tB_{n+1}\right)^{-1}\left(1+\tfrac{1}{2}(q-1)tB_{n+1}\right)K_n(;t)$$

The deformation coefficients satisfy recurrence relations in n

$$\frac{a_{n+1}(t)}{H_{n+1}}[-2R_{n+1}+(q-1)t\Gamma_{n+1}]+\frac{a_n(t)}{H_n}[2R_n+(q-1)t\Gamma_n] = -[x-b_n(t)]\frac{(q-1)t}{H_n}\Phi_n+2a_n(t)\left[\frac{R+(q-1)tS}{a_n(t)}-\frac{R-(q-1)tS}{a_n(qt)}\right]$$

$$\frac{a_{n+1}(qt)}{H_{n+1}}[2R_{n+1}+(q-1)t\Gamma_{n+1}]+\frac{a_n(qt)}{H_n}[-2R_n+(q-1)t\Gamma_n] = -[x-b_n(qt)]\frac{(q-1)t}{H_n}\Phi_n+2a_n(qt)\left[\frac{R+(q-1)tS}{a_n(t)}-\frac{R-(q-1)tS}{a_n(qt)}\right]$$

Final Compatibility

The spectral matrix $A_n(x;t)$ and the deformation matrix $B_n(x;t)$ satisfy

$$\begin{aligned}
&\left(1-\tfrac{1}{2}(q-1)xA_n(x;qt)\right)^{-1}\left(1+\tfrac{1}{2}(q-1)xA_n(x;qt)\right)\\
&\qquad\times\left(1-\tfrac{1}{2}(q-1)tB_n(x;t)\right)^{-1}\left(1+\tfrac{1}{2}(q-1)tB_n(x;t)\right)\\
&\qquad=\left(1-\tfrac{1}{2}(q-1)tB_n(qx;t)\right)^{-1}\left(1+\tfrac{1}{2}(q-1)tB_n(qx;t)\right)\\
&\qquad\qquad\times\left(1-\tfrac{1}{2}(q-1)xA_n(x;t)\right)^{-1}\left(1+\tfrac{1}{2}(q-1)xA_n(x;t)\right)
\end{aligned}$$

Define

$$\left(1-\tfrac{1}{2}(q-1)xA_n\right)^{-1}\left(1+\tfrac{1}{2}(q-1)xA_n\right)=\frac{1}{W+(q-1)xV}\begin{pmatrix}\mathfrak{W}_+ & -\mathfrak{T}_+\\ \mathfrak{T}_- & \mathfrak{W}_-\end{pmatrix}$$

$$\left(1-\tfrac{1}{2}(q-1)tB_n\right)^{-1}\left(1+\tfrac{1}{2}(q-1)tB_n\right)=\frac{1}{R+(q-1)tS}\begin{pmatrix}\mathfrak{R}_+ & -\mathfrak{P}_+\\ \mathfrak{P}_- & \mathfrak{R}_-\end{pmatrix}$$

Over-determined Linear System

Evaluation at zeros of $(W^2 - (q-1)^2 x^2 V^2)(x; t, qt)$ with respect to x
At $x = t$

$$r_{1-} t + r_{0-} + \frac{\mathfrak{W}_+(t; qt)}{\mathfrak{T}_+(t; qt)} p_+ = 0$$

whilst at $x = q^{-\gamma} t$

$$r_{1-} q^{-\gamma} t + r_{0-} + \frac{\mathfrak{W}_+(q^{-\gamma} t; qt)}{\mathfrak{T}_+(q^{-\gamma} t; qt)} p_+ = 0$$

at $x = q^{-1} t$

$$r_{1+} t + r_{0+} + \frac{\mathfrak{W}_-(q^{-1} t; t)}{\mathfrak{T}_+(q^{-1} t; t)} p_+ = 0$$

and at $x = q^{-\gamma-1} t$

$$r_{1+} q^{-\gamma} t + r_{0+} + \frac{\mathfrak{W}_-(q^{-\gamma-1} t; t)}{\mathfrak{T}_+(q^{-\gamma-1} t; t)} p_+ = 0$$

$D_5^{(1)}$ Painlevé system

One solution for component p_+ is

$$(\kappa_1 - q^{-1}\kappa_2)r_{1-}a_n(qt) = \left[-q^{-1}\kappa_1 + \frac{\mathfrak{z}_+(qt)}{(y(qt)-t)(y(qt)-q^{-\gamma}t)}\right]p_+$$

Here $\kappa_1 = q^{\alpha+\beta+\gamma+2+n}/t$, $\kappa_2 = q^{2-n}/t$

$$\Theta_n(y) = 0, \quad \mathfrak{z}_\pm = \frac{\nu}{y} \pm (q-1)\mu$$

where $\nu = (2W_n - W)(y)$, $\mu = (\Omega_n + V)(y)$.
A second solution for component p_+ is

$$(q\kappa_1 - \kappa_2)r_{1+}a_n(t) = \left[-\kappa_2 + \frac{\mathfrak{z}_-(qt)}{(y(t)-q^{-1}t)(y(t)-q^{-\gamma-1}t)}\right]p_+$$

Combine each of the solutions with

$$(q^{-1}\kappa_1 - \kappa_2)p_+ = (q\kappa_1 - \kappa_2)[r_{1+}a_n(t) - q^{-2}r_{1-}a_n(qt)]$$

Employ change of variables

$$\mathfrak{z}_+ = \frac{(y-q^{-1}t)(y-q^{-\gamma-1}t)}{\kappa_1 z}, \quad \mathfrak{z}_- = \kappa_1(y-q^{-1})(y-q^{-\beta-1})z$$

Deduce the q-recurrence relation [Jimbo & Sakai 1996]

$$\kappa_1\kappa_2 z(qt)z(t) = \frac{(y(t)-q^{-1}t)(y(t)-q^{-\gamma-1}t)}{(y(t)-q^{-1})(y(t)-q^{-\beta-1})}$$

Painlevé-Sakai Scheme and Askey Table Correspondence

- regular D_q-semi-classical weights imply a connection preserving property,
$$P(x;t) = Y_0(x,t)^{-1} Y_\infty(x;t)$$
with $P(x;qt) = P(x;t)$.
- Does this construction work for all of the Askey Table?
- identification with known integrable equations? Or do you get novel equations?
- characterisations of the solutions by q-difference and difference equations,
- solutions as non-linear generalisations of basic hypergeometric functions,
- have examples of classical solutions to q-Painlevé systems,
- require analogues of the local expansions for τ-functions about $t = 0, \infty$ and parameterisation of the connection matrices,
- what type of new averages in random matrix theory, determinantal point processes, mathematical physics, arise? Kaneko's q-integral and other q-Selberg integrals.