Superbosonization meets Free Probability

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Introduction

'Free probability' introduced by D. Voiculescu (1986) in the study of von Neumann algebras.

Gives calculational scheme by which to handle invariant ensembles of $N \times N$ random matrices, $N = \infty$.

Large-N limit of density of states encoded in Voiculescu R-transform: $R(k) = \sum_{n=0}^{\infty} c_{n+1} k^n$.

Free cumulants c_n are additive under addition of independent random matrices (for $N \to \infty$).

Voiculescu's analytical approach : define R-transform by inverting $z \mapsto g(z)$ (average trace of resolvent).

Combinatorial description of free cumulants in terms of non-crossing partitions given by R. Speicher (1994).

Free probability theory has not yet produced results for spectral correlation functions in the microscopic limit.

Method of commuting and anti-commuting variables (Wegner, Efetov) : results for *correlation functions*

(e.g., level statistics of small metallic grains,

localization in thick disordered wires,

scaling exponents at the Anderson transition, etc.)

Traditional variant (Hubbard-Stratonovich transformation) limited to *Gaussian* random variables.

Recent variant called 'superbosonization' allows to treat much wider class of distributions.

$H = H^*$ linear operator on Hermitian vector space $\mathbb{C}^N$.

Gaussian integral over commuting variables $\varphi \in \mathbb{C}^N$:

$$\mathrm{Det}^{-1}(z - H) = \int \mathrm{e}^{-(\tilde{\varphi}, \varphi z - H\varphi)}, \ \tilde{\varphi} = -\mathrm{i}\,\mathrm{sign}(\mathrm{Im}\, z)\, \overline{\varphi} \in (\mathbb{C}^N)^*$$

Gaussian ($\textit{Berezin}$) integral over anti-commuting variables ψ : $\quad \mathrm{Det}(w - H) = \int \mathrm{e}^{(\tilde{\psi}, \psi w - H\psi)}$.

$$\left\langle \frac{\mathrm{Det}(w - H)}{\mathrm{Det}(z - H)} \right\rangle_\mu = \int \Omega\left(\varphi \otimes \tilde{\varphi} + \psi \otimes \tilde{\psi}\right) \mathrm{e}^{-z(\tilde{\varphi}, \varphi) + w(\tilde{\psi}, \psi)},$$

Characteristic function : $\quad \Omega(K) = \int \exp\left(\mathrm{Tr}\, HK\right) d\mu(H)$.

From moments to cumulants

Commutative case $(N=1)$:

Moments $m_n = \int x^n d\mu(x)$ are generated by the characteristic function $\Omega(k) = \int e^{kx} d\mu(x) = \sum_{n=0}^{\infty} m_n \frac{k^n}{n!}$

The logarithm $\omega(k) = \ln \Omega(k) = \sum_{n=1}^{\infty} c_n \frac{k^n}{n!}$

generates the cumulants $c_n = \frac{d^n}{dk^n} \omega(k) \Big|_{k=0}$

Moments are expressed in terms of cumulants,

$$m_n = \frac{d^n}{dk^n}\Omega(k)\Big|_{k=0} = \frac{d^n}{dk^n}\mathrm{e}^{\omega(k)}\Big|_{k=0} = \sum_{p\in\Pi(n)}\prod_l c_l^{\nu_l(p)},$$

by summing over partitions $p \in \Pi(n): \ \sum_{l\geq 1} l\,\nu_l(p) = n$

where $\nu_l(p)$ is the number of blocks of length l.

Example $(n=8)$: $p = \{136\}\cup\{28\}\cup\{45\}\cup\{7\}$

$$\nu_1(p)=1,\quad \nu_2(p)=2,\quad \nu_3(p)=1,\quad \nu_4(p)=\ldots=0.$$

From moments to cumulants (III): R-transform

Probability measure $d\mu_N(H)$ for $N \times N$ matrices $H = H^*$

Characteristic function: $\Omega(K) = \int \mathrm{e}^{\mathrm{Tr}\, HK}\, d\mu_N(H)$

Moments $m_{n,N} = N^{-1} \int \mathrm{Tr}(H^n)\, d\mu_N(H)$ are generated by

differentiation: $m_{n,N} = N^{-1} \sum_{i_1,\dots,i_n} \frac{\partial^n}{\partial K_{i_n i_{n-1}} \cdots \partial K_{i_2 i_1} \partial K_{i_1 i_n}} \Omega(K) \Big|_{K=0}$

Let $m_n := \lim_{N \to \infty} m_{n,N}$

By definition, $z \mapsto g(z) = \sum_{n=0}^{\infty} m_n\, z^{-n-1}$ is inverted by

$k \mapsto k^{-1} + R(k)$ where $R(k) = \sum_{n=0}^{\infty} c_{n+1}\, k^n$ (R-transform).

R-transform (examples)

$$z = k^{-1} + R(k) \quad \Leftrightarrow \quad k = g(z)$$

Example 1: $R(k) = k$ (GUE)

$$g(z) = \frac{z}{2}\left(1 - \sqrt{1 - \frac{4}{z^2}}\right)$$ (Wigner semicircle)

Example 2: $R(k) = \dfrac{k}{1-k^2}$

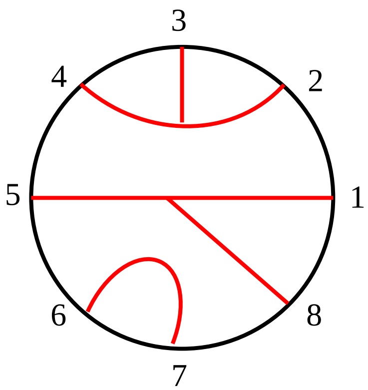
$$g(z) = \mathrm{i}\left(\sqrt{\frac{1}{27} - \frac{1}{4z^2}} - \frac{\mathrm{i}}{2z}\right)^{1/3} - \mathrm{i}\left(\sqrt{\frac{1}{27} - \frac{1}{4z^2}} + \frac{\mathrm{i}}{2z}\right)^{1/3}$$

Lueck, Sommers, MZ: J. Math. Phys. **47** (2006) 103304

Recall $R(k)=\sum c_{n+1}\, k^{n}$.

R. Speicher (1994): Combinatorial definition of ***free*** cumulants c_n by $m_n = \sum_{p\in \mathrm{NC}(n)} \prod_{l} c_l^{\nu_l(p)}$ where the sum runs over ***non-crossing*** partitions $p \in \mathrm{NC}(n)$.

Example $(n=8)$: $p = \{158\} \cup \{234\} \cup \{67\}$



Free cumulants ***add*** under convolution of measures (or addition of independent random matrices).

Large-N characteristic function

by free probability

Cumulants (non-commutative case)

Note: $\omega_\infty(K) = \lim_{N\to\infty} N^{-1} \ln \Omega(NK)$ is additive under addition of independent random matrices.

Assume $d\mu_N(H) = \mathrm{e}^{-N\,\mathrm{Tr}\,V(H)} dH$. Then $\omega_\infty(K) = \omega_\infty(g^{-1}Kg)$ for $g \in \mathrm{GL}_N$ and cumulants must be of the general form [where $\gamma_n(\pi)$ is constant on conjugacy classes]

$$\frac{\partial^n}{\partial K_{i_1 j_1}\,\partial K_{i_2 j_2}\cdots\partial K_{i_n j_n}}\,\omega_\infty(K)\bigg|_{K=0} = \sum_{\pi\in S_n}\gamma_n(\pi)\prod_l \delta_{i_l, j_{\pi(l)}}$$

Graphical methods suggest large-N hypothesis:

$\gamma_n(\pi) = c_n$ if π irreducible cycle, and $\gamma_n(\pi) = 0$ else.

Heuristics from planar graphs (I)

Recall $\Omega(NK) = \int e^{-N\,\mathrm{Tr}\,V(H) + N\,\mathrm{Tr}\,HK}\, dH$.

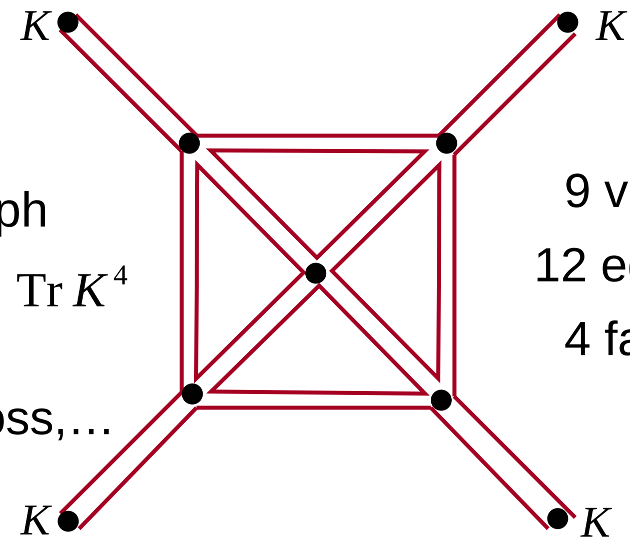
Perturbation theory for $\ln \Omega(NK)$ (connected graphs)

leads to topol. expansion: $\ln \Omega(NK) = \sum_{\chi=1,0,-1,\ldots} N^{\chi} \omega_{\chi}(K)$.

Leading contribution comes from summing all planar graphs (Euler characteristic $\chi = 1$).

Example: graph contributing to $\mathrm{Tr}\, K^4$

t'Hooft, Witten, Gross,…

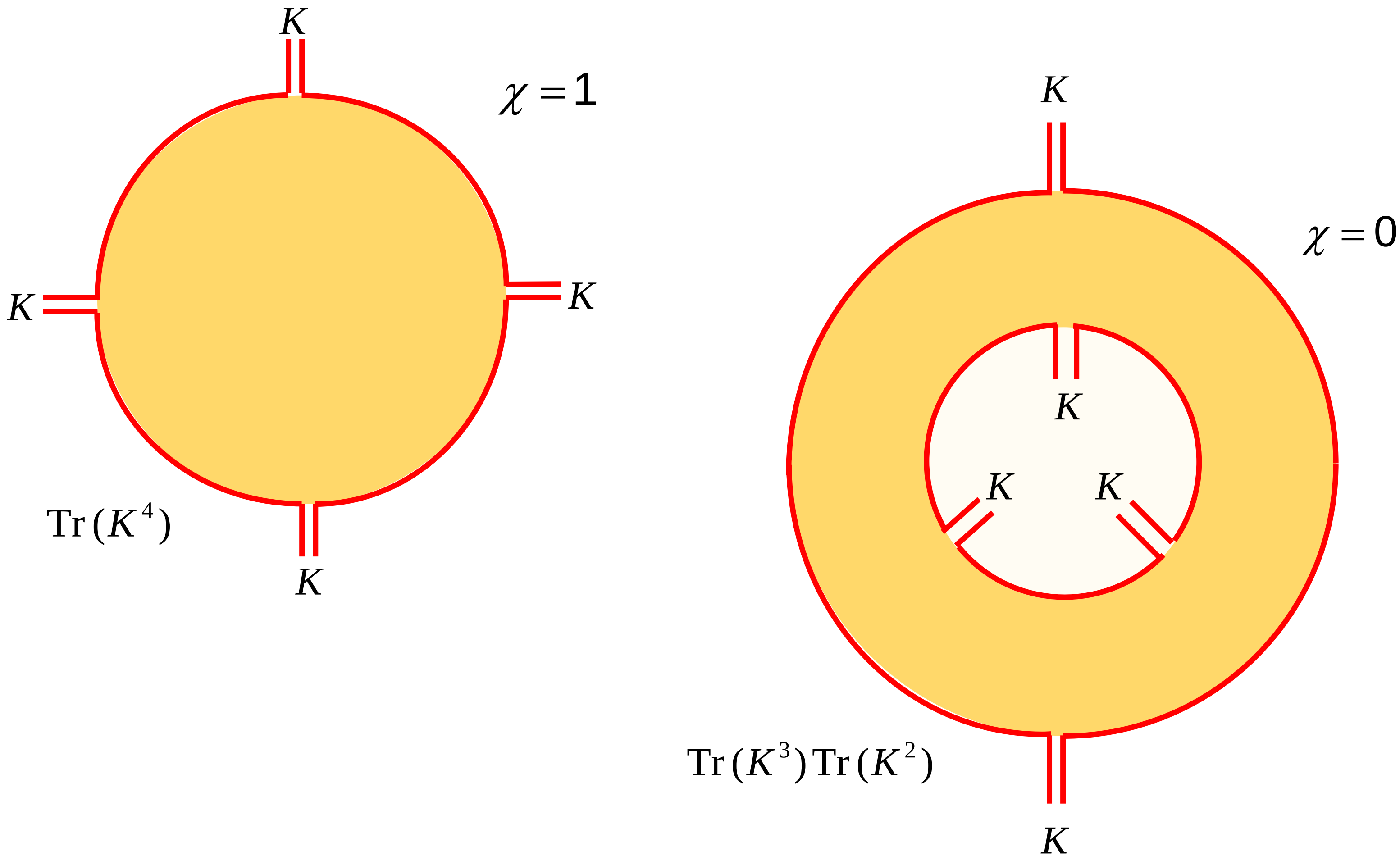


9 vertices: N^9

12 edges: N^{-12}

4 faces: N^4

Heuristics from planar graphs (II)



Large-N characteristic function from free probability (I)

Recall the large-N scenario from planar graphs:

$$\frac{\partial^n}{\partial K_{i_1 j_1} \cdots \partial K_{i_n j_n}} N^{-1} \ln \Omega(NK) \Big|_{K=0} \xrightarrow{N\to\infty} c_n \sum_{\pi\in[\mathrm{irr}]} \prod_l \delta_{i_l, j_{\pi(l)}}$$

By Speicher's combinatorial description the numbers c_n are identified as the free cumulants. In fact, taking $N \to \infty$ in the formula for the moment, we have $m_{n,N} =$

$$\sum_{i_1,\dots,i_n} \frac{N^{-n-1}\partial^n}{\partial K_{i_n i_{n-1}} \cdots \partial K_{i_1 i_n}} e^{\ln \Omega(NK)} \Big|_{K=0} \xrightarrow{N\to\infty} \sum_{p\in \mathrm{NC}(n)} \prod_l c_l^{\nu_l(p)}$$

Summary.

$$\omega_\infty(K) := \lim_{N\to\infty} N^{-1} \ln \Omega(NK).$$

For ensembles with GL_N-symmetry

we have $\omega_\infty(K) = \sum_{n=0}^{\infty} c_{n+1} \frac{\mathrm{Tr}(K^{n+1})}{n+1}.$

Note: the derivative of $\omega_\infty(K)$ for $K = k\Pi$

(rank-1 projector Π) is the R-transform:

$$\frac{d}{dk}\omega_\infty(k\Pi) = \sum_{n=0}^{\infty} c_{n+1} k^n = R(k)$$

Superbosonization

Hackenbroich, Weidenmüller (95)

Lehmann, Saher, Sokolov, Sommers (95)

Barruto, Brower, Svetitsky (01)

Efetov, Schwiete, Takahashi (04)

Guhr (06); Basile, Akemann (07)

Bunder, Efetov, Kravtsov, Yevtushenko, MZ (07)

Littelmann, Sommers, MZ (08)

Reminder: supersymmetry method

Characteristic function $\Omega(K) = \int e^{\operatorname{Tr} HK} d\mu(H)$

is evaluated on $K_{ij} = \sum_{a=1}^{p} \varphi_{i,a} \tilde{\varphi}_{a,j} + \sum_{b=1}^{q} \psi_{i,b} \tilde{\psi}_{b,j}$.

Generating function for spectral correlation functions:

$$\int D_{\varphi,\tilde{\varphi};\psi,\tilde{\psi}} \, f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) \equiv \int f$$

where the integral is along $\tilde{\varphi}_{a,j} = -\mathrm{i}\,\operatorname{sign}(\operatorname{Im} z_a)\, \overline{\varphi}_{j,a}$ and

$$f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = \Omega(K)\, \exp\left(-\sum_i \left(\sum_a \varphi_{i,a} z_a \tilde{\varphi}_{a,i} + \sum_b \psi_{i,b} w_b \tilde{\psi}_{b,i}\right)\right).$$

If $d\mu(H)$ invariant by some group G acting by conjugation

$H \mapsto gHg^{-1}$, then $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = f(g\varphi,\tilde{\varphi}g^{-1};g\psi,\tilde{\psi}g^{-1})$.

Let $p = 1,\ q = 0$ and consider GL_N-invariant holom. fctn

$$f : \mathbb{C}^N \times (\mathbb{C}^N)^* \to \mathbb{C},\ \ f(\varphi, \tilde{\varphi}) = f(g\varphi, \tilde{\varphi}\, g^{-1}),\ \ g \in \mathrm{GL}_N .$$

Fact (invariant theory): there exists a holomorphic function $F : \mathbb{C} \to \mathbb{C}$ such that $F(\langle \tilde{\varphi}, \varphi \rangle) = f(\varphi, \tilde{\varphi})$.

By push forward of the integral one has

$$\int_{\mathbb{C}^N} f(\varphi, \varphi^*)\, d^{2N}\varphi = c_N \int_{\mathbb{R}_+} F(r)\, r^{N-1} dr \quad \text{(if integral exists).}$$

generalization to p > 1: see Fyodorov, Nucl. Phys. B **621** (2002) 643

Superbosonization (special case: Grassmann variables only)

$p = 0$, $q = 1$. Let $F : \mathbb{C} \to \mathbb{C}$ be holomorphic.

Anticommuting variables $\psi = (\psi_1, \ldots, \psi_N)$.

Berezin integral $\int F(\langle \tilde{\psi}, \psi \rangle)\, d\tilde{\psi}\, d\psi \;:=$

$$= \frac{\partial^2}{\partial \psi_1 \partial \tilde{\psi}_1} \cdots \frac{\partial^2}{\partial \psi_N \partial \tilde{\psi}_N} F(\tilde{\psi}_1 \psi_1 + \ldots + \tilde{\psi}_N \psi_N)$$

$= F^{(N)}(0)$ (the N^{th} derivative at the origin)

$$= N! \oint_{\mathrm{U}(1)} F(z)\, z^{-N-1} dz / 2\pi \mathrm{i}.$$

q > 1: Kawamoto and Smit, Nucl. Phys. B **192** (1981) 100

Recall $f(\varphi,\tilde{\varphi};\psi,\tilde{\psi}) = f(g\varphi,\tilde{\varphi}\, g^{-1}; g\psi,\tilde{\psi}\, g^{-1})$

for $g \in G$. Let $G = \mathrm{GL}_N$ or $G = \mathrm{O}_N$ or $G = \mathrm{Sp}_N$.

Superbosonization exploits this symmetry to make a step of *reduction*:

The integral over $\varphi,\tilde{\varphi},\psi,\tilde{\psi}$ of the G-invariant function f is converted to an integral over a Riemannian symmetric superspace.

(The large number N of variables $\varphi,\tilde{\varphi},\psi,\tilde{\psi}$ then becomes a *parameter* of the integral.)

Let $G = \mathrm{GL}_N$.

Lift $f(\varphi, \tilde{\varphi}; \psi, \tilde{\psi})$ to $F(Q)$:

$$f(\varphi, \tilde{\varphi}; \psi, \tilde{\psi}) = F\begin{pmatrix} \langle \tilde{\varphi}, \varphi \rangle & \langle \tilde{\varphi}, \psi \rangle \\ \langle \tilde{\psi}, \varphi \rangle & \langle \tilde{\psi}, \psi \rangle \end{pmatrix}.$$

Theorem (Littelmann, Sommers, MZ). If $N \geq p$ and f holomorphic and Schwartz along $\tilde{\varphi} = -\mathrm{i}\overline{\varphi}^{\mathrm{t}}$, then

$$\int_{\tilde{\varphi} = -\mathrm{i}\overline{\varphi}^{\mathrm{t}}} f = \int_M DQ \; \mathrm{SDet}^N(Q) \; F(Q)$$

with integration domain $M \cong (\mathrm{GL}_p(\mathbb{C}) / \mathrm{U}_p) \times \mathrm{U}_q$ and $\mathfrak{gl}$-invariant Berezin integration form DQ.

Large-N limit: saddle analysis

Average resolvent by superbosonization:

$$g(z) = \lim_{N\to\infty} N^{-1} \int \mathrm{Tr}\,(z-H)^{-1} d\mu_N(H) =$$

$$\lim_{N\to\infty} \int_M DQ\ \mathrm{e}^{N(-z\,\mathrm{STr}\,Q + \omega_\infty(Q) + \mathrm{STr}\ln Q)}\, Q_{00}\,.$$

$$Q = \begin{pmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{pmatrix}, \quad M = \mathbb{R}_+ \times \mathrm{U}_1$$

Taking $N \to \infty$ gives saddle-pt eqn for $Q = k\,\mathrm{Id}$:

$z = R(k) + k^{-1}$. Solving for k yields $k = g(z)$.

Thus, superbosonization at the saddle-point level reproduces Voiculescu's formula relating the average trace of resolvent to the R-transform.

$\Omega(K)\big|_{K=\varphi\tilde{\varphi}+\psi\tilde{\psi}}$ has $\mathrm{gl}_{p|q}$-supersymmetry.

So does $\hat{\Omega}(NQ)\,\mathrm{SDet}^{N}(Q)\,\mathrm{e}^{-Nz\,\mathrm{STr}\,Q}$.

Large-N correlation functions result from integrating $\mathrm{e}^{-\mathrm{STr}(Q\delta z)}$ over saddle-point manifold $(=\mathrm{GL}_{p|q}$-orbit).

Universality classes $\leftrightarrow$ types of $\mathrm{GL}_{p|q}$-orbit.

Story similar for $G=\mathrm{GL}_N, \mathrm{O}_N, \mathrm{Sp}_N$

$$\frac{d}{dk}\left(k^{-1}+R(k)\right)\Big|_{k=g(z)} \begin{cases} \neq 0: & \text{bulk universality (W/D)} \\ = 0: & \text{edge universality (Airy)} \end{cases}$$

Free probability theory provides the proper framework in which to take the large-N limit of the density of states for ensembles which are invariant but non-Gaussian.

Free cumulants are the Taylor coefficients of the (logarithm of the) characteristic function which is encountered when using superbosonization.

The group of supersymmetries determines the critical integration manifold (saddle points).