Bosonization for random matrices and electron systems in arbitrary dimensions.

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Collaborations with:

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Bosonization: mapping of electron models onto a model describing collective excitations (charge, spin excitations, diffusion modes, etc).

Origin of the word: 1D electron systems.

The main idea: writing the electronic operators ψ as

$$\psi \propto \exp(i\varphi)\exp\left(i\int \rho dx\right)$$

A simple Hamiltonian H

$$H = \int [K\rho^2 + N\left(\nabla\varphi\right)^2]dx \qquad \left[\rho,\varphi\right] = -i$$

ρ -Density fluctuations operator

K-compressibility, N-average density

Importance of long wave length excitations!

Bosonization: for Tomonaga-Luttinger model (long range interaction) (Luttinger, Tomonaga (196?))

The most general form conjectured by K.E. & A. Larkin (1975)

Microscopic theory Haldane (1982)

....

Disordered systems (without interaction)

$$F_{dif}[Q] = \frac{\pi\nu}{8} Str \int [D(\nabla Q)^2 + i(\omega + i\delta)\Lambda Q] d\mathbf{r}$$

$$Q^2 = 1$$

D is the diffusion coefficient, Q are supermatrices

- Diffusive σ-model-success in describing disordered systems with short range defects (replica-Wegner 1979, Efetov, Larkin, Khmelnitskii 1980, supersymmetry-Efetov 1982)

Zero-dimensional σ -model.

$$F_0[Q] = \frac{\pi\nu V(i\omega + i\delta)}{8} Str \Lambda Q$$

V-volume

Three classes of universality: orthogonal, unitary and symplectic

(there are more (Zirnbauer, Altland))

Zero-dimensional σ -model from random matrices:
Verbaarschot, Weidenmuller, Zirnbauer (1984)

Wigner-Dyson statistics for quantum chaotic billiards
Bohigas, Giannoni, Schmit (1984)

Q. Can one extend the supersymmetry method to clean chaotic billiards and random matrices with an arbitrary correlations of entries?

Example:

$$P(H) = \exp\left(-\frac{1}{2}\sum_{i,j=1}^{M} c(|i-j|)^{-1} \sum_{a,b=1}^{n} H_{ij}^{ab} H_{ji}^{ba}\right)$$

where c(|i-j|) is an arbitrary function

(c(i-j)=const gives the Wigner-Dyson statistics)

Q. What about interacting electron systems in arbitrary dimensions?

A. A (local) field theory can be derived in all the cases!

Where is the problem in extending the derivation of the σ -model to the ballistic case of for arbitrary random matrices?
Two important steps in deriving the σ-model):

1. Singling out slow modes

$$\int W(r-r')\overline{\psi}(r)\psi(r)\overline{\psi}(r')\psi(r')dr \to$$

$$\sum_{p_1,p_2;q<q_0} W(q)\overline{\psi}_{p_1}\psi_{p_1+q}\overline{\psi}_{p_2}\psi_{p_2-q} + 2\sum_{p_1,p_2;q<q_0} W(p_1-p_2)\overline{\psi}_{p_1}\psi_{p_2}\overline{\psi}_{p_2-q}\psi_{p_1+q}$$

Makes a sense only for q< b^{-1}, b is the range of W(r-r').

No applicability for averaging over spectrum!

2. Saddle-point approximation (equivalent to self-consistent Born approximation)

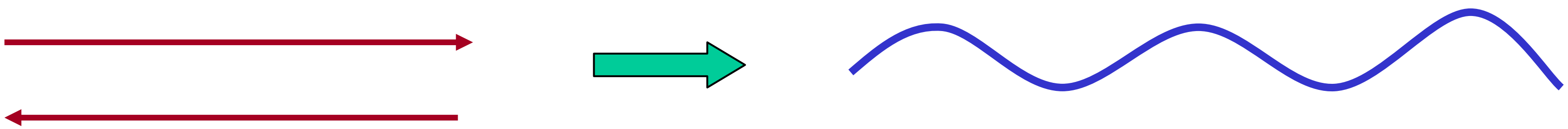
$$Q = <0|(-iH + Q/2\tau)^{-1}|0>$$

Also bad for a long range disorder!

The basic features of the suggested approach:
no singling out slow modes, no saddle-point approximation!
Instead: exact mapping to a field theory with supermatrices Q
-a (super) bosonization scheme.

What is done: Integration over electrons is replaced by integration over collective excitations (exactly).

Assumptions: no assumptions.



2 electrons

Diffuson(kineton)

Formulation of the problem
The Hamiltonian *H*

$$Z[J] = \int \exp(-\mathcal{L})\mathcal{D}(\psi, \bar{\psi}), \qquad (2)$$

$$\mathcal{L} = -i \int \bar{\psi}(\mathbf{r})\mathcal{H}_J(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}d\mathbf{r}', \qquad (3)$$

$$\mathcal{H}_J(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')\left(\hat{H} - \varepsilon + \frac{\omega + i\delta}{2}\Lambda\right) - J(\mathbf{r}, \mathbf{r}'), \qquad (4)$$

$\psi(r)$ are *8* - component supervectors

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} - \varepsilon_F + U(\mathbf{r}),$$

L - effective Lagrangian with sources

How to rewrite Z[J] in terms of an integral over supermatrices?

Exact mapping (no averaging over U(r) has been performed)

$$Z[J] = \int \exp(-F[Q])\mathcal{D}Q,$$
$$F[Q] = \frac{i}{2}\int \mathrm{Str}\left[\mathcal{H}_J(\mathbf{x}) * Q(\mathbf{x}) - i\ln Q(\mathbf{x})\right] d\mathbf{x}.$$

Superbosonization

Proof: equations for $\delta Z[J]/\delta J$ are the same (superstructure is important).

Wigner representation is made and **x=(r,p)**

$$A(\mathbf{x}) * B(\mathbf{x}) = A(\mathbf{x}) e^{\frac{i}{2}\left(\overleftarrow{\nabla}_{\mathbf{r}}\overrightarrow{\nabla}_{\mathbf{p}} - \overrightarrow{\nabla}_{\mathbf{r}}\overleftarrow{\nabla}_{\mathbf{p}}\right)} B(\mathbf{x})$$

* -Moyal product

The method resembles bosonization schemes in field theory: P.B. Wiegmann (1989), S.R. Das, A.Dhar, G. Mandal and S.R. Wadia (1992), D.V. Khveshchenko (1994). Some overlap with (Hackenbroich&Weidenmuller (1995), Fyodorov (2002)).

However, the supersymmetry simplifies everything and allows to do more!

Saddle point of the action at J=0: $q_0(\mathbf{x}) = i\mathcal{H}^{-1}_{J=0} = g(\mathbf{x})$

where $g(\mathbf{x})$ the one particle Green function (without sources)

Semiclassics: the potential $U(\mathbf{r})$ is smooth

⇓

Using the representation $Q(\mathbf{x}) = T(\mathbf{x}) * \bar{q}(\mathbf{x}) * \bar{T}(\mathbf{x})$,
assuming that T(x) is smooth but q is close to $g(\mathbf{x})$ we come to the integral

$$\nu \int g(\mathbf{x}) d\xi \simeq i\nu \int \frac{d\xi}{\xi(\mathbf{p}) + i\delta\Lambda} = ib + \pi\nu\Lambda$$

$$\xi(\mathbf{p}) = \mathbf{p}^2/2m - \varepsilon_{\mathrm{F}}$$

⇓

The modulus of the momentum is pinned to the Fermi surface: $\mathbf{p} = \mathbf{n}p_{\mathrm{F}}, \mathbf{n}^2 = 1$

Warning: In a clean billiards levels are discrete and the integration over p cannot be performed before an averaging over the spectrum!

This gives a ballistic σ -model valid at all distances exceeding the wave length λ_F

$$F[Q_{\mathbf{n}}] = \frac{\pi\nu}{2}\mathrm{Str}\int d\mathbf{r}d\mathbf{n}\Big[\Lambda\bar{T}_{\mathbf{n}}(\mathbf{r}) \times \left(v_{\mathrm{F}}\mathbf{n}\nabla_{\mathbf{r}} - p_{\mathrm{F}}^{-1}\nabla_{\mathbf{r}}U(\mathbf{r})\nabla_{\mathbf{n}}\right)T_{\mathbf{n}}(\mathbf{r}) + i\left(\frac{(\omega+i\delta)}{2}\Lambda - J_{\mathbf{n}}(\mathbf{r})\right)Q_{\mathbf{n}}(\mathbf{r})\Big],$$

$$Q_{\mathbf{n}} = T_{\mathbf{n}}\Lambda\bar{T}_{\mathbf{n}}.$$

No problem of "mode locking"!

Averaging has not been done but can be carried out easily for any potential U(**r**)! It is assumed that the sample is infinite for the above formula.

(Still a regularizer should be obtained)

3. Application to random matrices with non-trivial correlations.

Let H be a NxN random matrix.

The averages are introduced as:

$$\langle \ldots \rangle_H = C_N \int dH_{ij} (\ldots) \exp\left(-\frac{1}{2} \sum_{i,j} \left(C_{ij} \right)^{-1} H_{ij}^2 \right)$$

C_{ij} is arbitrary

The generating functional Z(J) can be reduced exactly to the form:

$$Z(J) = \int DQ \exp\left[-\frac{i}{2} Str \sum_{1<i<N} Q_i H'_i - \frac{1}{2} Str \sum_{1<i<N} \ln Q_i + \frac{1}{8} Str \sum_{1<i,j<N} Q_i C_{ij} Q_j \right]$$

$$H'_i = -\varepsilon + \frac{\omega + i\delta}{2} \Lambda - J_i$$

Q is 8x8 supermatrix, J is a source.

Z(J) describes a 1D "spin" chain with the interaction A_{ij}

Important feature: Q(i) depends on one variable i only!

Structure of the supermatrix Q

$$Q = \begin{pmatrix} A & \sigma \\ \rho & B \end{pmatrix}$$

σ, ρ -Matrices consisting of Grassmann variables

A, B -Matrices consisting of conventional numbers

Contour of integration over A and B

A is unitary

B is positive Hermitian (positive real eigenvalues)

Example: density of states of almost diagonal matrix

$$\rho(E) = (2\pi N)^{-1}\,\mathfrak{Im}\sum_{i=1}^{N} \mathrm{i}\langle \mathrm{Tr}\, Q_i \rangle$$

$$\langle \ldots \rangle = \int (\ldots)\exp(-F[Q])dQ$$

$$F[Q] = Str\left[-\frac{i}{2}\sum_{1<i<N} EQ_i + \frac{1}{2}\sum_{1<i<N} \ln Q_i - \frac{1}{8}\sum_{1<i,j<N} Q_i C_{ij} Q_j \right]$$

$$Q = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}$$

Expansion in off-diagonal terms $C_{ij}, i \neq j$

Zero order:

$$\rho_d(E) = \pi^{-1}\mathfrak{Im}\,\mathrm{i}\langle a_1 \rangle = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}Eb-(C_0/2)b^2}\,db = (2\pi C_0)^{-1/2}\mathrm{e}^{-E^2/2C_0}$$

The first correction:

$$\delta\rho(E) = -(4\pi N)^{-1}\Im\mathrm{m}\,\mathrm{i}\sum_i \sum_{j\neq k} C_{jk}\,\langle \mathrm{Tr}(Q_i)\,\mathrm{STr}(Q_j Q_k)\rangle$$

The final result

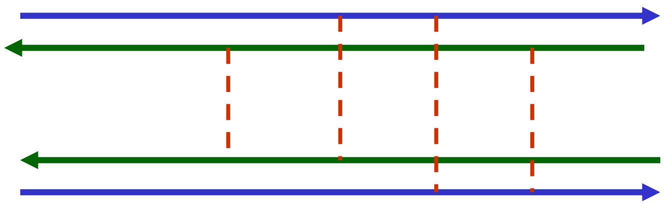
$$\delta\rho(E) = -\frac{C_1}{2C_0}\frac{d}{dE}\left(\mathrm{e}^{-E^2/C_0}\mathrm{erfi}(E/\sqrt{2C_0})\right)$$

where $C_1 = \sum_{j\neq 1} C_{1j}$

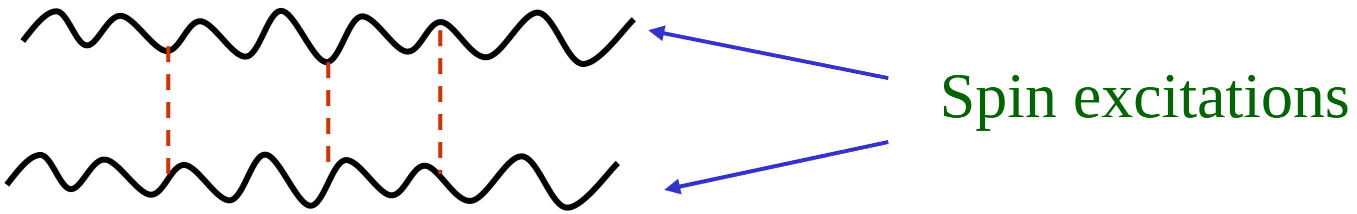
and $\mathrm{erfi}(x) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}$ Imaginary error function

Electron systems with interaction
(quasiclassical theory)

Interesting contributions come not from two electron correlations but from four electron ones!



Equivalent representation



Reduction from electrons to collective excitations: Bosonization

The scheme of the method.

1. Singling out slow varying $k < k_c \le p_F$ pairs in the interaction
2. Hubbard-Stratonovich transformation: reduction of the interaction to slowly varying effective fields.
3. Replacement of the fermions in the effective field by bosonic excitations (equations for quasiclassical Green functions).
4. Integration over the effective fields.

$$L_{\text{int}} = \frac{1}{2}\sum_{\sigma,\sigma'}\int \psi^*_\sigma(r)\psi^*_{\sigma'}(r')V(r-r')\psi_{\sigma'}(r')\psi(r)\,dr\,dr' \rightarrow$$

$$\frac{1}{2}\sum_{\sigma,\sigma'}\int dP_1 dP_2 dK\left\{V_2\chi^*_\sigma(P_1)\chi_\sigma(P_1+K)\chi^*_{\sigma'}(P_2)\chi_{\sigma'}(P_2-K) - V_1(\vec{p}_1-\vec{p}_2)\chi^*_\sigma(P_1)\chi_{\sigma'}(P_1+K)\chi^*_{\sigma'}(P_2)\chi_\sigma(P_2-K)\right\}$$

$$P = (\varepsilon, \vec{p})$$

$$K = (\omega, k)$$

Effective slow field Φ : 2x2 spin matrix

$$g_{\mathbf{n}}^{\Phi}(\mathbf{R},\tau,\tau') = i\int_{-\infty}^{\infty} G_{\mathbf{p}}^{\Phi}(\mathbf{R},\tau,\tau')\frac{d\xi}{\pi}$$

-quasiclassical Green function

$$\xi = \frac{p^2}{2m} - \varepsilon_F$$

Equation and constraint for g

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\tau'} - iv_F\mathbf{n}\nabla\right) g_{\mathbf{n}}^{\Phi}(\mathbf{R},\tau,\tau') + \left(g_{\mathbf{n}}^{\Phi}(\mathbf{R};\tau,\tau')\Phi_{\mathbf{n}}(\mathbf{R},\tau') - \Phi_{\mathbf{n}}(\mathbf{R},\tau) g_{\mathbf{n}}^{\Phi}(\mathbf{R};\tau,\tau')\right) = 0 \quad (2.37)$$

$$\Phi(x,\vec{n}) = i\varphi(x,\vec{n}) + \vec{\sigma}\vec{h}(x,\vec{n})$$

$$\int g_{\vec{n}}^{\Phi}(\tau-\tau_1) g_{\vec{n}}^{\Phi}(\tau_1-\tau') d\tau_1 = \delta(\tau-\tau')$$

$$\vec{p} = p_F\vec{n}$$

$$\vec{n}^2 = 1$$

$(x,\vec{n})$ -Coordinates in phase space

The solution

$$g^{\Phi}_{\mathbf{n}}(\mathbf{R},\tau,\tau') = T_{\mathbf{n}}(\mathbf{R},\tau)\, g_0(\tau-\tau')\, T^{-1}_{\mathbf{n}}(\mathbf{R},\tau')$$

⟺ Generalization of the Schwinger Ansatz

Equations for

$$M_{\mathbf{n}}(x) = \frac{\partial T_{\mathbf{n}}(x)}{\partial \tau} T^{-1}_{\mathbf{n}}(x)$$

$$M_{\mathbf{n}}(x) = \rho_{\mathbf{n}}(x) + \mathbf{S}_{\mathbf{n}}(x)\,\sigma$$

$S_n(r,\tau)$ Spin

$\rho_n(r,\tau)$ Charge

Equations for the charge and spin excitations: starting point for the calculations.

$$\left(-\frac{\partial}{\partial \tau} + i v_F \mathbf{n} \nabla_{\mathbf{R}}\right) \rho_{\mathbf{n}}(x) = -i \frac{\partial \varphi_{\mathbf{n}}(x)}{\partial \tau}$$

$$\left(-\frac{\partial}{\partial \tau} + i v_F \mathbf{n} \nabla_{\mathbf{R}}\right) \mathbf{S}_{\mathbf{n}}(x) + 2i\left[\mathbf{h}_{\mathbf{n}}(x) \times \mathbf{S}_{\mathbf{n}}(x)\right] = -\frac{\partial \mathbf{h}_{\mathbf{n}}(x)}{\partial \tau}$$

⟺

No interaction leading to infrared divergences.

High dimensional bosonization of Luther, Haldane, ……..

Effective interaction leading to divergent contributions at T=0 (logarithmic in any dimensions)

Effective field theory for the spin excitations.

Supervectors: $\psi(X)=\begin{pmatrix}\chi(X)\\ S(X)\end{pmatrix}$

$\chi(X)$ -anticommuting
$S(X)$ -commuting

Effective Lagrangian $L[\psi]$

$$Z=\int \exp\left(-\mathrm{L}\left[\psi\right]\right) D\psi$$

Z -Partition function, L-Lagrangian

$$\mathrm{L}\left[\psi\right]=\mathrm{L}_0\left[\psi\right]+\mathrm{L}_2\left[\psi\right]+\mathrm{L}_3\left[\psi\right]+\mathrm{L}_4\left[\psi\right]$$

$$\mathrm{L}_0\left[\psi\right]=-2i\nu\int\overline{\psi_\alpha\left(X\right)}H_0\psi_\alpha\left(X\right)dX$$

$$H_0=\begin{pmatrix}-iv_F\left(\mathbf{m}\nabla\right)\tau_3\Sigma_3 & -\partial/\partial\tau\\ -\partial/\partial\tau & -iv_F\left(\mathbf{m}\nabla\right)\tau_3\Sigma_3\end{pmatrix}$$

$L_0[\psi]$ -Lagrangian of free excitations

Interaction terms

$$L_4[\psi] = -8\nu\varepsilon_{\alpha\beta\gamma}\varepsilon_{\alpha\beta_1\gamma_1}\int\left(\overline{\psi_\beta(X)}\tau_3\psi_\gamma(X)u\right)$$
$$\times\tilde{\hat{\Gamma}}\left(u\overline{\psi_{\beta_1}(X)}\tau_3\psi_{\gamma_1}(X)\right)dX \qquad (4$$

$$\varepsilon_{\alpha\beta\gamma} = -\varepsilon_{\alpha\gamma\beta} = -\varepsilon_{\beta\alpha\gamma} = 1$$

$$L_3[\psi] = -8\nu\sqrt{2}i\varepsilon_{\alpha\beta\gamma}\int\left(\overline{\psi_\beta(X)}\tau_3\psi_\gamma(X)u\right)$$
$$\times\tilde{\hat{\Gamma}}\left(\bar{F}_0\overline{\partial_x(u)}\tau_3\psi_\gamma(X)\right)dX$$

$$L_2[\psi] = 4i\nu\int\left(\overline{\psi_\alpha(X)}\tau_3\overleftarrow{\partial}_x(u)F_0\right)$$
$$\times\tilde{\hat{\Gamma}}\left(\bar{F}_0\overline{\partial_x(u)}\tau_3\psi_\gamma(X)\right)dX$$

L_0, L_4 are supersymmetric

L_2, L_3 violate the supersymmetry ⇨ contribution to thermodynamics

Next step: renormalization group treatment: $\psi(X) = \psi_0(X) + \tilde{\psi}(X)$

$\psi_0(X)$ -fast

$\tilde{\psi}(X)$ -slow

Integration over the fast part $\psi_0(X)$

Sounds good but….

Short distances are not well defined, artificial cutoffs are necessary, curvature of the Fermi surface is not properly taken into account, everything is quite cumbersome.

Fortunately, an exact (quite compact) mapping using BRST-type supersymmetry is possible!

$$Z = Z_0 \int \exp(-S[\Psi]) D\Psi$$

Z_0-partition function of the ideal Fermi gas

$\Psi(X, X'; R)$ -(anticommuting) superfield

$X = \{r, \sigma\}$ -coordinate and spin

$R = \{\tau, u, \theta, \theta^*\}$ $\quad \tau$ -imaginary time, 0<u<1

θ, θ^* -anticommuting variables

$$S[\Psi] = S_0[\Psi] + S_2[\Psi] + S_3[\Psi] + S_4[\Psi]$$

$$S_0[\Psi] = -\frac{i}{4} \int [\Psi(X', X) H_{0,X,X'} \Psi(X', X) - c.c] dX dX' dR$$

$$H_{0,X,X'} = -\frac{\partial}{\partial \tau} - (\varepsilon(-i\nabla_X) - \varepsilon(-i\nabla_{X'}))$$

S_2, S_3, S_4 -interactions of the second, third and fourth order in $\Psi(X, X'; R)$

proportional to the bare interaction V

Conclusions.

Everything can be bosonized.

Hopes for new developments.