Replicas, Toda Lattice Equation and QCD

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I. Motivation

Miscellaneous Results

Physical Motivation

Some Miscellaneous Results

- ✓ Parametric correlations of energy levels with normalized energy difference r and an Aharonov-Bohm flux that differs by ϕ ,

$$\langle \delta\rho(-r/2,\phi/2)\delta\rho(r/2,\phi/2)\rangle = \frac{1}{2}\int_1^\infty d\lambda e^{-2\pi^3 g\phi^2\lambda^2 + i\pi r\lambda}\int_{-1}^1 d\lambda_1 e^{2\pi^3 g\phi^2\lambda_1^2 - i\pi r\lambda_1}$$

Altshuler-Simons-1993

For $\phi = 0$ this gives the two-point level correlation function of the Gaussian Unitary Ensemble. Efetov-1983, JV-Zirnbauer-1985.

- ✓ Density of resonances for M open channels with transmission coefficients T_a and broken time reversal invariance as a function of the complex energy $E + iy$,

$$\rho(E,y) = -\frac{1}{4\pi}\int_{-\infty}^\infty dk e^{-iky}\prod\frac{1}{2-ikT_a}\int_{-1}^1 d\lambda e^{-\lambda y}\prod_{a=1}^M (2+\lambda T_a).$$

Fyodorov-Sommers-1997

More Miscellaneous Results

- ✓ Spectral density in the complex plane for weak nonhermiticity

$$\rho_X(y) = \frac{e^{-2a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} d\lambda e^{-2\lambda^2 a^2 - iy\lambda} \int_0^1 dt \cosh(ty) e^{-a^2 t^2/2},$$

where a is the nonhermiticity parameter.

Fyodorov-Khoruzhenko-Sommers-1996

- ✓ The microscopic limit of the resolvent of the QCD Dirac operator for $z \in \mathcal{C}$ and $x = Nz$,

$$\frac{1}{x}\frac{d}{dx} xG(x) = \int_{-1}^{1} \frac{d\lambda_1}{\sqrt{\lambda_1^2 - 1}} e^{\lambda_1 x} \int_1^{\infty} \frac{d\lambda}{\sqrt{\lambda^2 - 1}} e^{-x\lambda}.$$

Damgaard-Osborn-Toublan-JV-1998

Comments

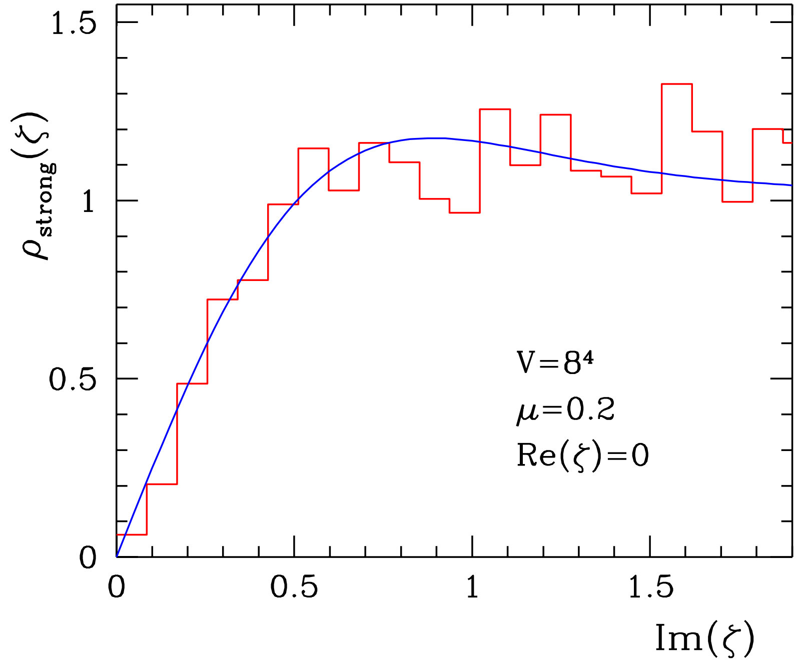
All the above results were obtained by means of the super-symmetric method.

Results factorize into the product a compact and a noncompact integral.

This factorization only takes place for $\beta = 2$.

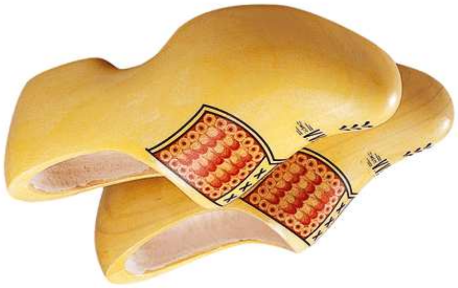
The corresponding partition functions should have an additional structure that is not explained in the framework of the supersymmetric method.

Physical Motivation



Histogram of the real part of Dirac eigenvalues obtained from lattice QCD simulation at nonzero chemical potential. Akemann-Wettig-2004

- √ The spectral density of the QCD Dirac operator at nonzero chemical in the neighborhood of zero is given by nonhermitean random matrix theory.
- √ A first analytical results was obtained in the limit $|\zeta| \gg 1.$ Akemann-2003
- √ Although the fit seems to work reasonably well, it fails at small ζ .
- √ We need a result that is valid when $|\zeta| \gg 1$ is not satisfied.



II. The Replica Trick

The R-Trick and its Failure

Kanzieper's Approach

Alternative Approach

The R-Trick

Quenched averages can be calculated from the replicated partition function

$$Z_n(z) = \langle \det{}^n (H + z) \rangle$$

by taking the limit $n \to 0$ after differentiation with respect to source terms. For example, the resolvent is obtained from

$$\lim_{n \to 0} \frac{1}{n} \frac{d}{dz} \log Z_n(z)$$

If Z_n is an analytical function of n including the positive real axis and a region around zero, the replica limit can be evaluated if we know the partition function for all positive integer values of n .

Failure of the R-Trick

The r-trick trivially gives correct perturbative results, such as an the coefficients of an expansion in $1/N$.

In 1984 Martin and I showed that the r-trick gave different nonperturbative results for bosonic and fermionic replicas and both results were wrong.

Starting with our paper I strongly believed that the r-trick was doomed to fail at the nonperturbative level.

Work by Kamenev and Mézard, by and Yurkevich and Lerner and by Zirnbauer, all in 1999, only strengthened my conviction that the r-trick was a failed approach.

The paper by Eugene came as a complete shock to me. I had not the slightest doubt that the paper ought be wrong and set out to show that this was indeed the case.

Together with Kim Splittorff we convinced ourselves that Eugene was right after all, and in doing so we found an alternative formulation of the replica trick.

Approach of Kanzieper

- √ The random matrix partition function satisfies a Toda lattice equation.
- √ Because of this, the corresponding τ -function satisfies a Painlevé equation where n appears as a coefficient.
- √ The desired result follows from the solution of this Painlevé equation after taking the limit $n \to 0$.
- √ This differential equation has qualitatively different solutions for $n > 0$ and $n < 0$ with the solution for $n = 0$ coinciding with the exact quenched random matrix result.

Kanzieper-2002

Kanzieper-Osipov-2002

Alternative Approach

The replica trick works if the partition is analytic in n along the positive real axis including a region around zero.

However, there is another possibility: If partition functions for different values of n are related by a recursion relation, we may be able to evaluate the replica limit without relying on analyticity in n .

We will now show that, for $\beta = 2$, invariant random matrix ensembles generically obey such recursion relation.

Splittorff-JV-2003

III. Toda Lattice Equation

Average Characteristic Polynomials of Invariant Nonhermitean Random Matrix Ensembles

Hankel Form of the Average Characteristic Polynomial

Sylvester Identity and Toda Lattice Equation

General Invariant Nonhermitean RMT

An invariant nonhermitean random matrix partition function of ensemble of complex random matrices W can be expressed in terms of an integral over eigenvalues z_k of W ,

$$Z_N = \int \prod_{i=1}^{N} d^2 z_i w(z_i, z_i^*) |\Delta_N(z)|^2$$

We take the weight function $w(z, z^*)$ symmetric under the exchange of z and z^* . Then the corresponding orthogonal polynomials, $P_k(z)$, can be chosen to have real coefficients.

The nonhermiticity of the random matrix ensemble is characterized by the nonhermiticity parameter μ . Results for Hermitian random matrix theories can be obtained by taking the limit $\mu \to 0$. This might be easier than working directly with hermitian theories.

The basic object that can be constructed out of the orthogonal polynomials is the kernel

$$K_{N+n}(z_l, z_m^*) = \sum_{i=0}^{N+n-1} P_i(z_l) P_i(z_m^*).$$

n-Flavor Partition Functions

Theorem: (Akemann-Vernizzi-2003)

$$Z_n(z_k, z_k^*) = \left\langle \prod_{k=1}^{n} \det(z_k - W) \det(z_k^* - W^\dagger) \right\rangle = \frac{\det_{1\le l,m\le n} K_{N+n}(z_l, z_m^*)}{\Delta_n(\{z_k\})\Delta_n(\{z_k^*\})}$$

The reason for this theorem is that a random matrix partition function corresponds to free fermions with the eigenvalues interpreted as the positions of particles. Wiegman

Next we take the limits $z_k \to z$ and $z_k^* \to z^*$ and denote the partition function by $Z_n(z, z^*)$. Then the resolvent and spectral density are given by

$$\begin{aligned} G(z, z^*) &= \mathrm{Tr}\frac{1}{z+W} = \lim_{n\to 0} \frac{1}{n}\partial_z \log Z_n(z, z^*), \\ \rho(z, z^*) &= \sum_k \delta(z - z_k) = \frac{1}{\pi} \lim_{n\to 0} \frac{1}{n}\partial_z \partial_{z^*} \log Z_n(z, z^*). \end{aligned}$$

Hankel Form of Characteristic Polynomial

The limit $z_k \to z$ can be taken by Taylor expanding the z_k about z .

After multiple subtractions of rows and columns, Vandermonde determinants can be factorized from $\det K_{N+n}(z_l, z_m^*)$. These determinants cancel the Vandermonde determinants in the denominator

The derivative of the Taylor expansion combine into a Hankel form for the expectation value of the characteristic polynomial

$$Z_n(z, z^*) = \det[\delta_z^k \delta_{z^*}^m K_{N+n}(z, z^*)]_{0 \le l, m \le n-1}.$$

Differentiating a determinant is equivalent to differentiating with respect to all rows or with respect to all columns.

Therefore co-factors can be expressed as derivatives of $Z_{n-1}(z, z^*)$.

Toda Lattice Equation

Sylvester identity for a matrix A with co-factors C_{ij} and double co-factors $C_{ij,pq}$

$$C_{ij}C_{pq} - C_{iq}C_{pj} = C_{ij,pq}\det A,$$ Forrester-book.

If we apply this to $i = j = n-2$ and $p = q = n-1$ of $Z_{n-1}(z,z^*)$ we obtain

$$\delta_z\delta_{z^*}\log Z_n(z,z^*) = \frac{\pi n}{2}(zz^*)^2\frac{Z_{n+1}(z,z^*)Z_{n-1}(z,z^*)}{Z_n^2(z,z^*)}.$$

After taking the replica limit we find

$$\rho(z,z^*) = \frac{1}{2}zz^*Z_{n=1}(z,z^*)Z_{n=-1}(z,z^*),$$ Splittorff-JV-2003.

This shows that the spectral density factorizes into the product of a bosonic and a fermionic partition function.

The two-point correlation function can be obtained in the same way and is also expressed as a product of a fermionic and bosonic partition function.

With a minor modification, the Toda lattice equation is also valid for finite N .

Akemann-Osborn-Splittorff-JV-2004

Graded Toda Lattice Equation

The Toda lattice equation can be extended to ratios of characteristic polynomials.

Then deerivatives of the partition function can be obtained by putting $n = 0$ rather than taking the limit $n \to 0$.

Splittorff-JV-2004

IV. Applications of the Toda Lattice Equation

Nonhermitean Chiral Random Matrix Theories

Result for the Spectral Density of the Quenched Dirac Operator

Parametric Correlations

Complex Eigenvalue Distributions

Examples of Random Matrix Theories

Dirac operator of QCD at nonzero chemical potential (Osborn-2006)

$$D(\mu) = \begin{pmatrix} 0 & iW + \mu Y \\ iW^\dagger + \mu Y^\dagger & 0 \end{pmatrix},$$

with both W and Y distributed according to an invariant probability distribution.

The generating function for the Dirac spectrum is given by

$$Z_n(z, z^*) = \langle \det{}^n(D + z) \det{}^n(D^\dagger + z^*) \rangle.$$

This partition function satisfies the Toda lattice equation.

In the microscopic limit

$$zN = \text{fixed}, \qquad \mu^2 N = \text{fixed} \quad \text{for} \quad N \to \infty$$

the Toda lattice equation and its solution become universal.

Universal Form of Partition Function at $\mu \neq 0$

In the microscopic domain, the random matrix partition function has a universal form given by Toublan-JV-2000

$$Z_n^{\rm micro}(z,z^*) = \int_{U(n)} dU \exp[GN{\rm Tr}(MU^\dagger + M^\dagger U) - FN{\rm Tr}[U,B][U^\dagger,B]].$$

mass matrix
diag(z, z^*)

charge matrix containing the chemical potentials

This partition function satisfies the Toda lattice equation.

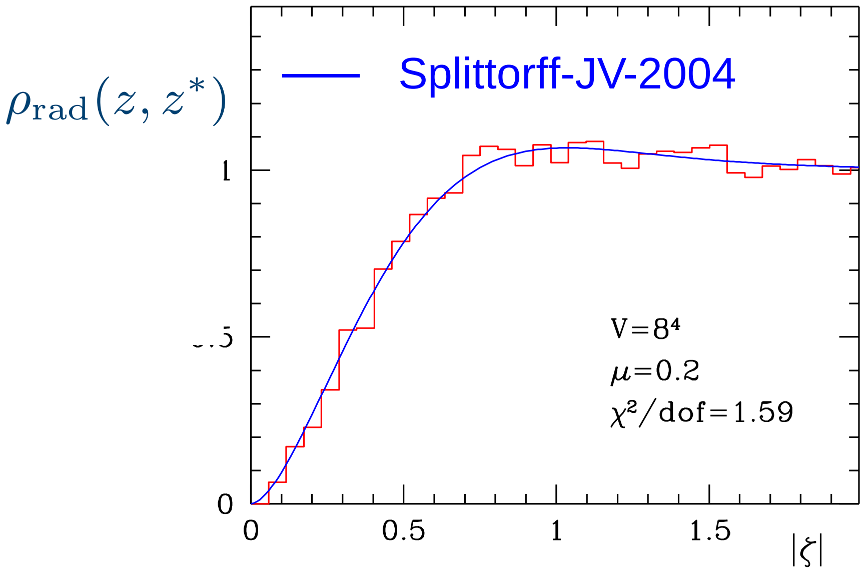
The partition function with D given by

$$D = \left(\begin{array}{cc} 0 & iW + \mu\mathbf{1} \\ iW^\dagger + \mu\mathbf{1} & 0 \end{array} \right),$$

has the same universal limit in the microscopic domain.

Quenched Lattice Dirac Spectra at $\mu \neq 0$

Comparison of lattice Dirac spectra to analytical results. Splittorff-JV-2004



Wettig-2004

Radial distribution ρ_{rad} of Dirac eigenvalues

The analytical result has been obtained from the random matrix representation of the static part of the chiral Lagrangian exploiting the Toda lattice hierarchy in the flavor index.

$$\rho(z, z^*) = \frac{1}{2} z z^* Z_1^{\mathrm{micro}}(z, z^*) Z_{-1}^{\mathrm{micro}}(z, z^*).$$

Parametric Correlations

Because $\det(D^\dagger(\mu)+z^*) = \det(D(-\mu)+z)$ for imaginary μ the generating function $Z_n(z,z^*)$ becomes the generating function for parametric correlations.

Because it is an invariant random matrix partition function, it satisfies the Toda lattice equation, and the two-point parametric correlation function can be obtained by taking the replica limit of the Toda lattice equation.

This way results obtained by means of the supersymmetric method are recovered.

Altland-Simons-1993, Weidenmüller-2004

Parametric correlations for the QCD Dirac operator have been derived from the Toda lattice equation as well.

Damgaard-Heller-Splittorff- Svetitsky-2006, Akemann-Damgaard-Osborn-Splittorff-2008

Complex Eigenvalue Distributions

The eigenvalue distribution of the QCD partition function at nonzero chemical potential is given by

$$\rho(z, z^*) = \langle \det(D(\mu) + m) \sum_k \delta^2(z - \lambda_k) \rangle.$$

Because $D(\mu)$ is nonhermitean, the eigenvalue distribution is complex. Using the replica trick the eigenvalue distribution can be written as

$$\rho(z, z^*) = \lim_{n \to 0} \frac{1}{n} \partial_z \partial_{z^*} \langle \det(D(\mu) + m) \sum_k |\det(D(\mu) + z)|^{2n} \rangle.$$

This partition function can again be expressed as a Hankel determinant and therefore satisfies a Toda lattice equation.

Toda Lattice Equation and Complex Eigenvalue Distribution

After taking the replica limit of the Toda lattice equation, the spectral density is expressed in terms of partition functions

$$\rho(m,z,z^*) = \frac{zz^*}{2}\frac{\langle \det(D(\mu)+m)|\det(D(\mu)+z|^2\rangle\langle \det(D(\mu)+m)|\det(D(\mu)+z|^{-2}\rangle}{\langle \det(D(\mu+m))\rangle^2}.$$

Akemann-Osborn-Splittorff-2004

This result agrees with the result obtained from complex orthogonal polynomials. Osborn-2004

Partition functions are extensive in the thermodynamic limit. To obtain a finite spectral density, free energies have to cancel.

Let us first look at the phase quenched partition function

V. Interpretation of Nonhermitean Dirac Spectra

Random Matrix Theory and Goldstone Bosons

Spectra of Nonhermitean Partition Functions in Terms of Phases of Partition Functions

Absence of Phase Transitions for Bosonic Partition Functions

Phase Diagram of the One-Flavor QCD Dirac Operator at Nonzero Chemical Potential

Random Matrix Theory and Goldstone Modes

- √ The flavor (replica) symmetry of the RMT generating function is broken spontaneously.
- √ In the microscopic limit universal random matrix results can be expressed as an integral over the Goldstone manifold.
- √ The mass of the Goldstone modes is given by

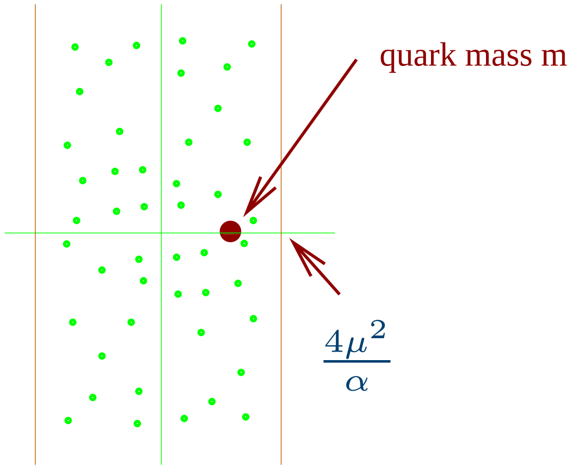
$$m_G^2 = \alpha(z + z^*).$$

- √ Real or imaginary chemical potentials become interesting in the microscopic limit at a scale equal to m_G
- √ As a reminder, the universal form of the microscopic limit of random matrix partition function at nonzero chemical potential (or Aharonov-Bohm flux) is given by

$$Z_n^{\rm micro} = \int_{U(n)} dU \exp[\frac{1}{2} V\Sigma {\rm Tr}(MU^\dagger + M^\dagger U) - \frac{1}{4} F^2 {\rm Tr}[U,B][U^\dagger,B]].$$

Dirac Spectrum of Phase Quenched Theory

- √ Because $\det(D^\dagger(\mu)+m)=\det(D(-\mu)+m)$ the chemical potential in the phase quenched partition function has the quantum numbers of the charged Goldstone bosons.
- √ The phase quenched partition function undergoes a phase transition to a Bose condensed phase at $\mu=m_G/2$.
- √ This has to be the point where the quark mass hits the boundary of the eigenvalues (Gibbs-1986).



Scatter plot of Dirac eigenvalues

$$m_G^2=\alpha m=4\mu^2,\quad m=\frac{1}{2}(z+z^*)$$

For small enough chemical potential, the width, of course, increases linearly with μ .

Phase Quenched Partition Function

The phase quenched partition function has two phases: a normal phase when the chemical potential is less than the mass of the Goldstone bosons, and a bose condensed phase when the chemical potential is larger than the mass of the Goldstone bosons.

Because the spectral density is normalized and positive definite, the only possibility is that in the domain of the eigenvalues, the free energies of the fermionic and the bosonic partition function will cancel.

A phase transition takes place when leaving the support of the spectrum. The fermionic partition function undergoes a phase transition while the bosonic partition function does not. Therefore the free energies are no longer balanced and we necessarily find that the spectral density is exponentially suppressed with the volume.

Partition function with One Flavor

When spectral density is complex we have the possibility that the the free energies do not have to cancel for the spectral density to remain normalized.

The partition function $Z(m, z, z*)$ has three different quark masses. This implies that we have two different types of charged Goldstone bosons with mass $\sim \sqrt{z + z*}$ and with mass $\sim \sqrt{m + z*}$.

Therefore was can have three different phases: a normal phase, a condensed phase of zz^* bosons and a condensed phase of mz^* bosons.

The bosonic partition function does not have a phase transition. We expect that in the phase with the zz^* condensate the bosonic and fermionic free energies cancel as for the phase quenched partition function.

We expect that a transition to a mz^* condensate takes place when $m < \mathrm{Re} z$. Since the free energy of this phase is less negative than the free energy of the zz^* phase we conclude that the spectral diverges exponentially with the volume in this phase.

To maintain the normalization of the spectral density the spectral density has to oscillate with a period $\sim 1/V$.

Why Does the $n = -1$ Partition Function Have No Phase Transition

$$Z_{N,n=-1} = \langle \prod_k d^2 z_k w(z_k, z_k^*) \frac{\prod_{k>l} |z_k - z_l|^2}{\prod_k |z - z_k|^2}.$$

The integral over z_k is logarithmically divergent because of the contribution of a single eigenvalue, z_N , close to z .

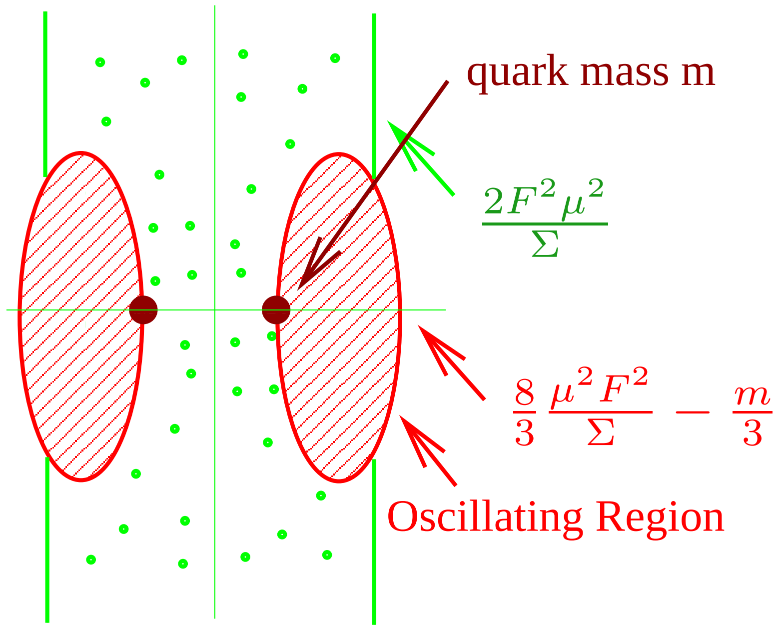
Because of the Vandermonde determinant, the probability of finding two or more eigenvalues close to z is not divergent.

For or $z = z_N$ the denominator cancels against the factors in the Vandermonde containing z_N . We thus find (Splittorff-JV-2006)

$$Z_{N,n=-1} = \log \epsilon \frac{w(z, z^*)}{|z|^2} Z_{N-1,n=0}.$$

Because $Z_{N-1,n=0}$ is the quenched partition function is does not depend on z and z^* , and does not have a phase transition as as function of these parameters. (see Splittorff-JV-Zirnbauer-2008

Phase Diagram of Dirac Spectrum



Dirac spectrum for Full QCD.

Generating function for Dirac spectrum

$$Z = \langle \det(D+\mu\gamma_0+m)|\det(D+\mu\gamma_0+z)|^2\rangle.$$

Three flavor partition function with an isospin and strangeness chemical potential.

Kogut-Toublan-2001

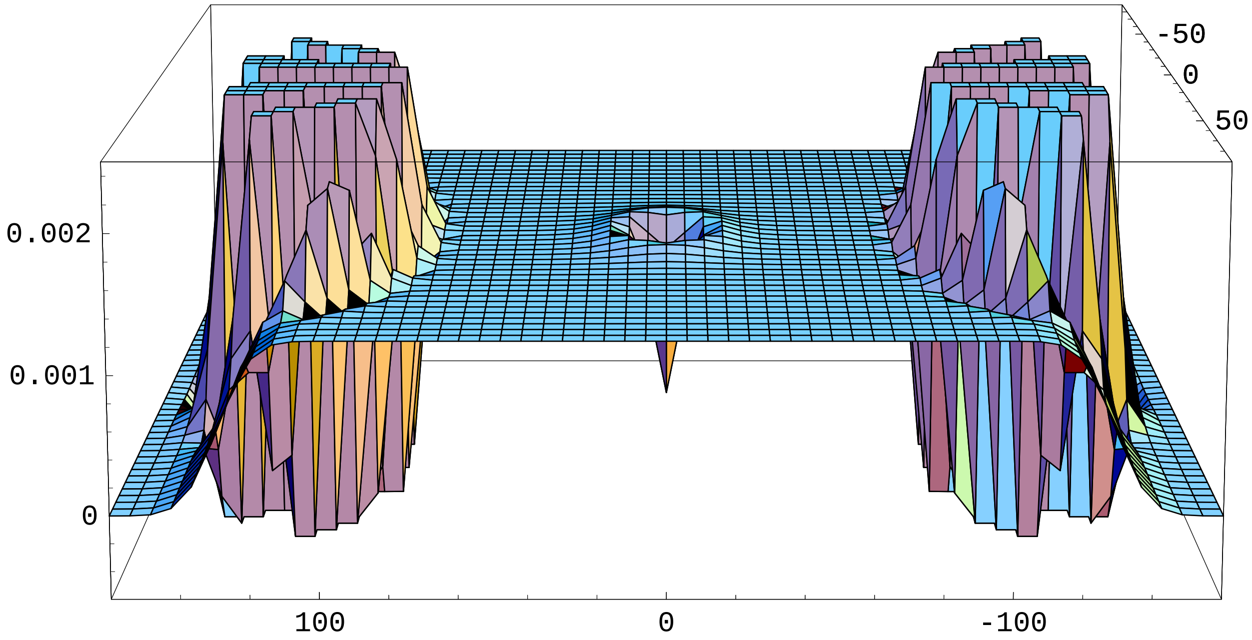
The dotted region is in a pion condensed phase.

The dashed region is in a kaon condensed phase.

The remainder of the complex phase is in the normal phase.

Osborn-Splittorff-JV-2005/2008, Osborn-2004

QCD at $\mu \neq 0$



Real part of the spectral density for QCD with one flavor at nonzero chemical potential.

The oscillatory region is responsible for the discontinuity in the chiral condensate.

Osborn-Splittorf-JV-2005/2008, Osborn-2004

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Conclusions

VI. Conclusions

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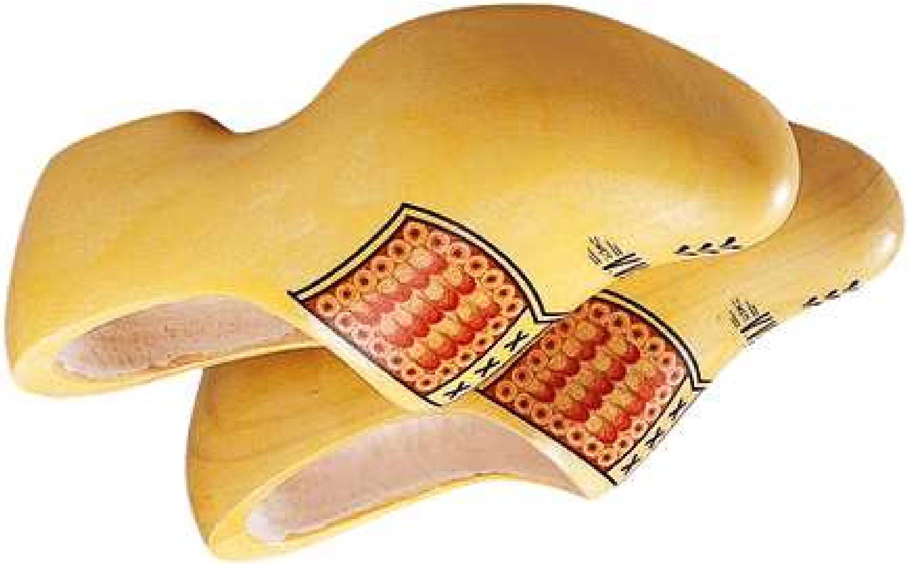
- ✓ New results were obtained for the microscopic limit of the Dirac spectrum at nonzero chemical potential.
- ✓ The behavior of Dirac spectra could be interpreted in terms of phases of partition functions.
- ✓ We showed that a class of bosonic partition function at nonzero chemical potential do not have a phase transition.
- ✓ These results are important for explaining chiral symmetry breaking in QCD at nonzero chemical potential.



Topics

- ✓ Quenching
- ✓ Homogeneity of Dirac Spectrum
- ✓ Ultraviolet Cancellations
- ✓ Phase Diagram: Development of two scales
- ✓ Smooth transition from the microscopic limit to the thermodynamic limit
- ✓ QCD at small chemical potentials follows from RMT
- ✓ Two condensates

Conclusions