Yad Hashmona 29.3.09

Stochastic Growth Models and Matrix-valued Diffusions

Herbert Spohn

TUM

1. KPZ equation for growth

2. Line ensembles

   A] matrix-valued diffusions

   (Dyson 1962)

   B] growth processes

   (Johansson 2002)

3. Flat initial conditions

1.) Kardar, Parisi, Zhang 1986

height function $h(x,t) \in \mathbb{R}$, 1D

$$\frac{\partial}{\partial t} h = \lambda \left(\frac{\partial}{\partial x} h\right)^2 + \nu_0 \frac{\partial^2}{\partial x^2} h + \eta$$

white noise

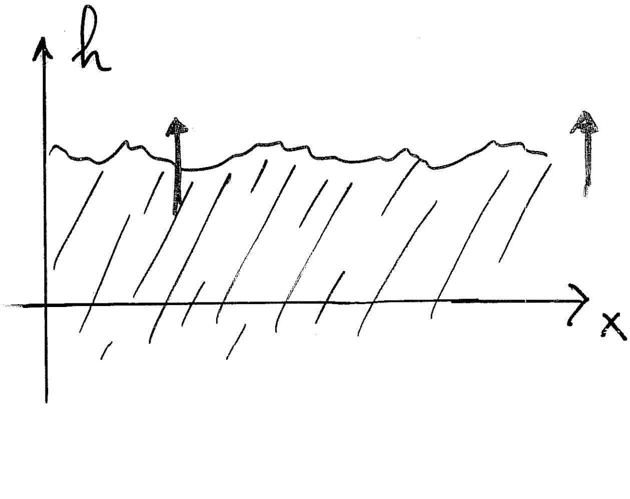
$\nu_0 > 0$

$\langle \eta \rangle = 0$

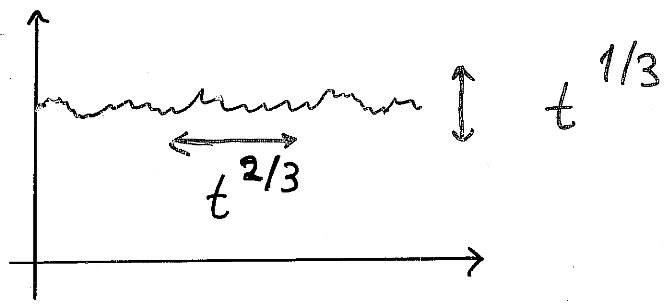
$\langle \eta(x,t)\, \eta(x',t') \rangle = \Gamma \delta(x-x')\delta(t-t')$

initial values $h(x, t=0)$

- moving interface (relative frame)



- $\lambda \left(\frac{\partial}{\partial x} h\right)^2$ slope dependent growth velocity
- $\nu_0 \frac{\partial^2}{\partial x^2} h$ smoothening
- η random detachment/atachment



- lattice regularization $\qquad$ slope $u = \frac{\partial}{\partial x} h$

$$\frac{d}{dt} u_j = -\frac{1}{2}\lambda \left(u_{j+1}^2 + u_{j+1} u_j - u_j u_{j-1} - u_{j-1}^2 \right)$$

$$+ \nu_0 \left(u_{j+1} - 2u_j + u_{j+1} \right) + \underbrace{\eta_{j+1} - \eta_j}_{\text{independent white noise}}$$

- invariant measures are

  $\{ u_j, j \in \mathbb{Z} \}$ independent Gaussians

  variance $\quad$ fixed

  mean $\quad$ arbitrary

  (slope is conserved)

- NO detailed balance

• discretized KPZ from Katzav, Schwartz, PRE 2004

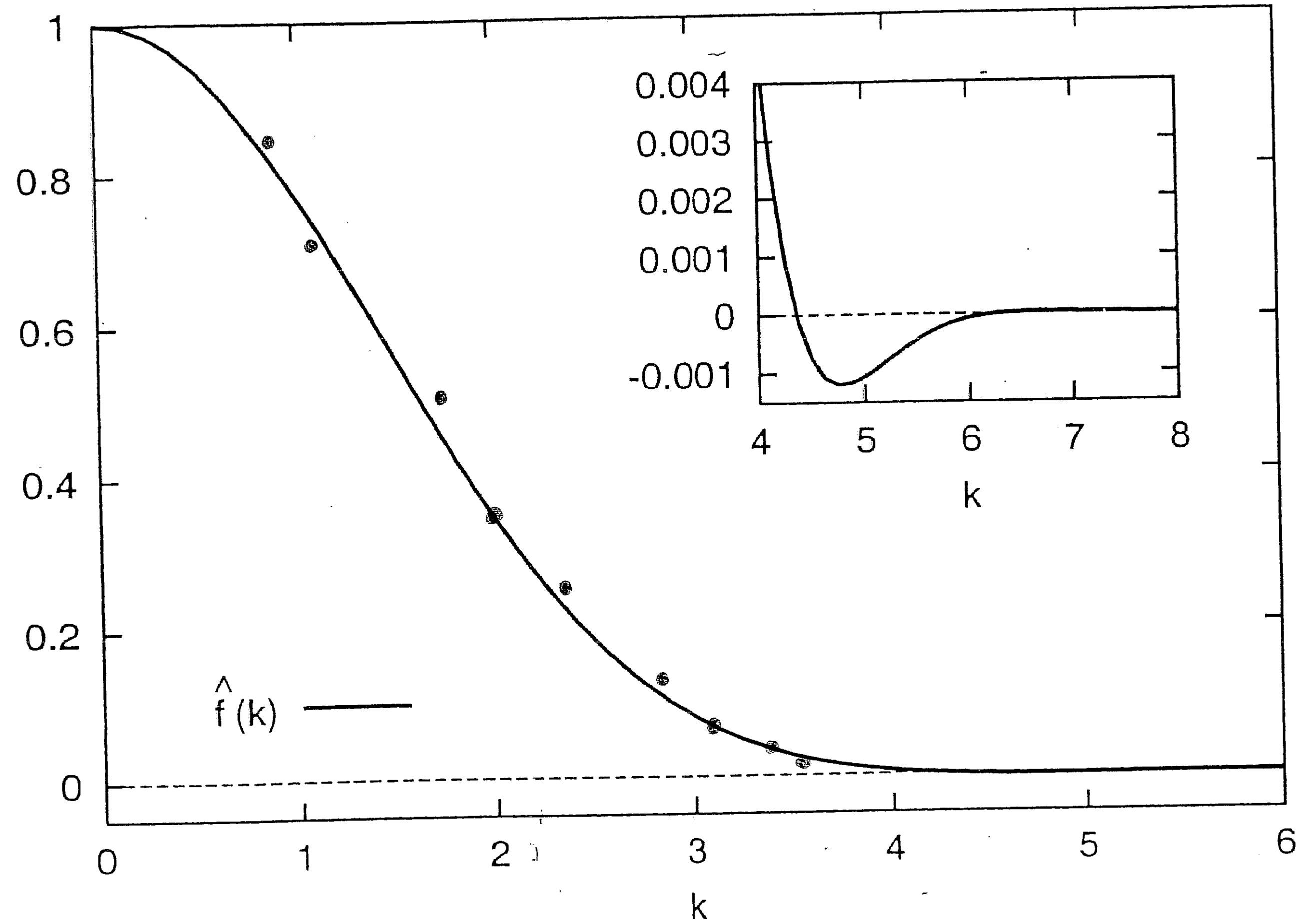


Figure 4: *The Fourier transform $\widehat{f}(k)$ of the scaling function f*
exact solution

experimental realization?

- ballistic deposition

inconclusive

- $\lambda = 0$ (linear)

$h(0,t)$ Gaussian variance $\sqrt{t}$

(Arratia)

Bechinger et. al. (Stuttgart) (2004)

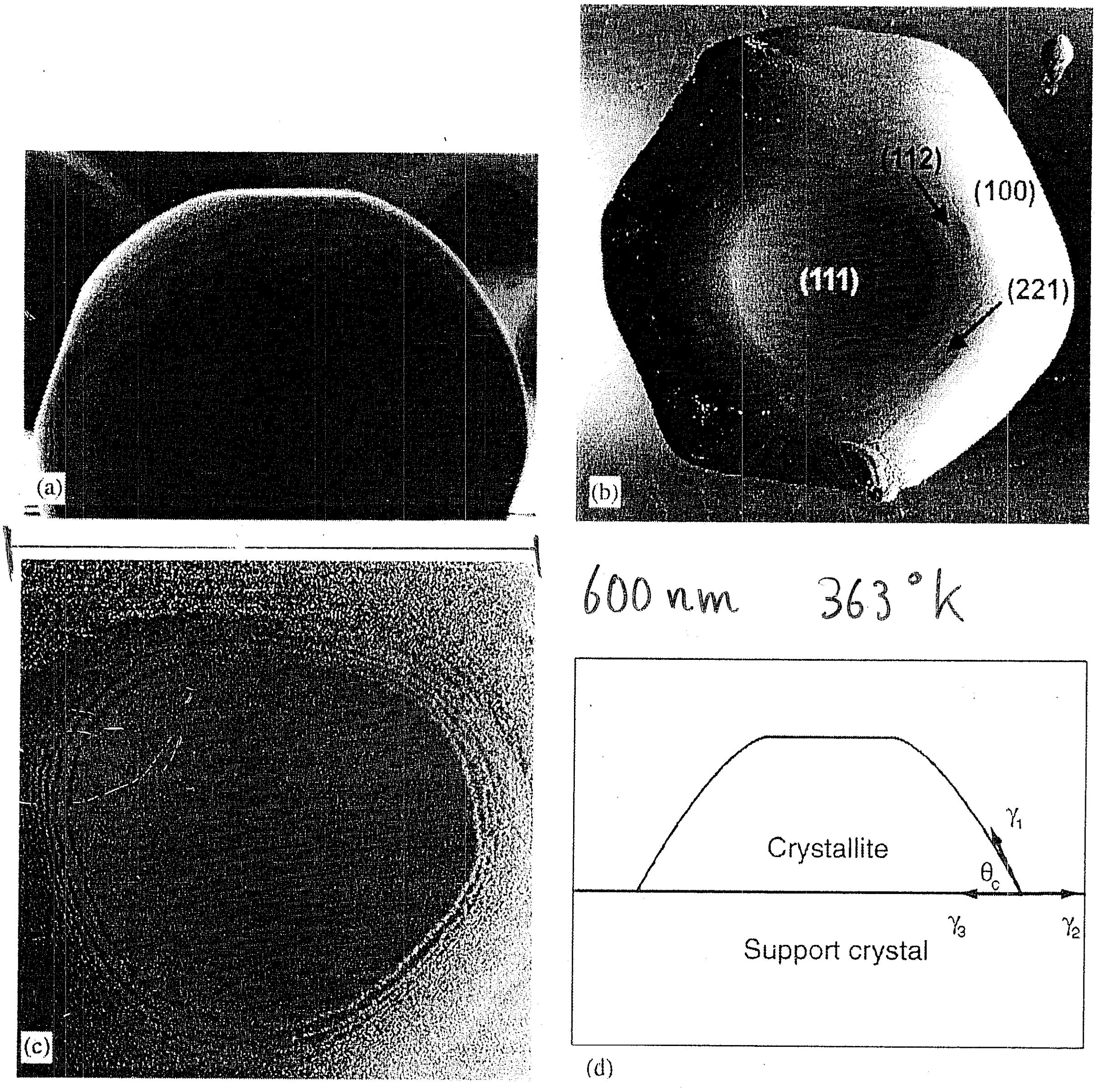
- Einstein, Williams et. al. (Maryland) (2007)

3D crystals in equilibrium

fluctuations of facet edge

from RM

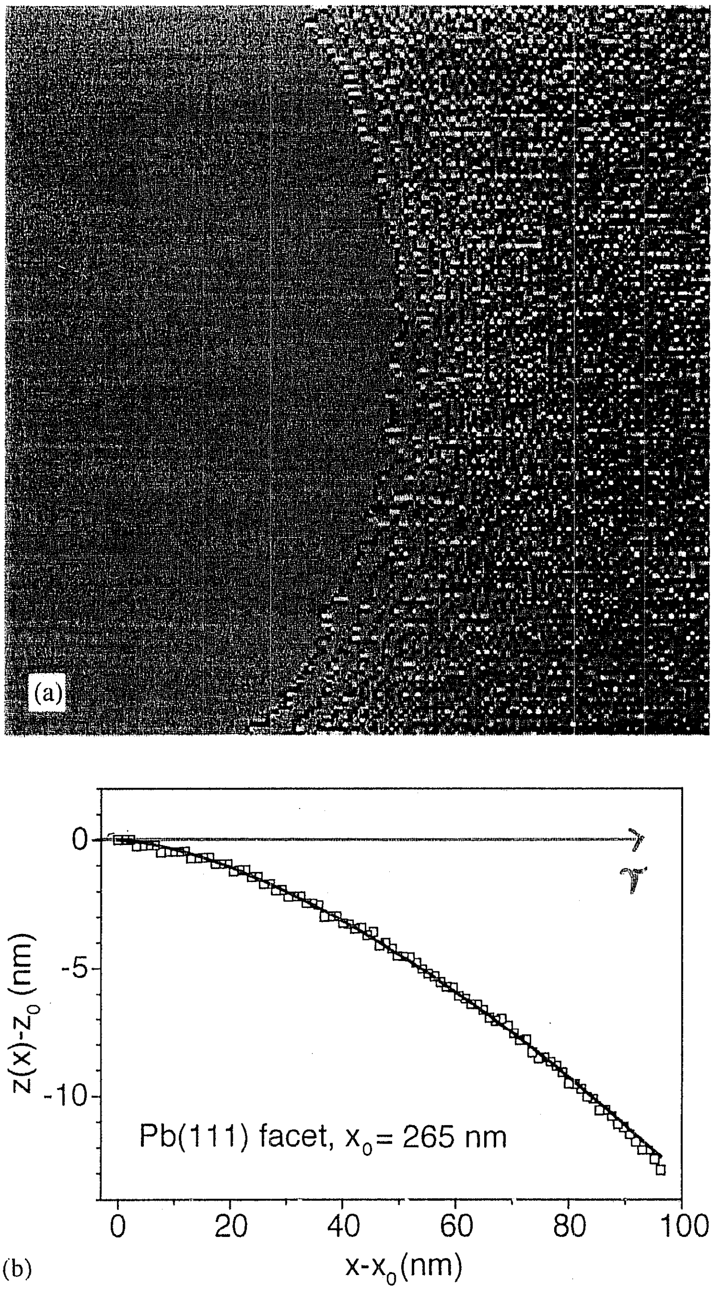
Pb on Ru



ig. 2. (a) SEM image of an equilibrated Pb crystallite viewed along a [1$\bar{1}$0] direction, annealed at $T = 473$ K and imag
: room temperature [37]. (b) STM image of Pb crystallite on Ru(001) at 323 K, showing top (111) facet, average fac
.dius ~ 140 nm, and smaller side facets [49]. (c) STM image of a 3D Pb crystallite at 363 K, showing (111) facet
.e top (average facet radius ~ 230 nm) and step resolved vicinal surface next to this main facet. (d) Schematic of a 3
ystallite supported by a flat substrate. Note definitions of contact angle, surface and interface free energies.

Bonzel, Phys. Rep. 385 (2003)

Pb on Ru



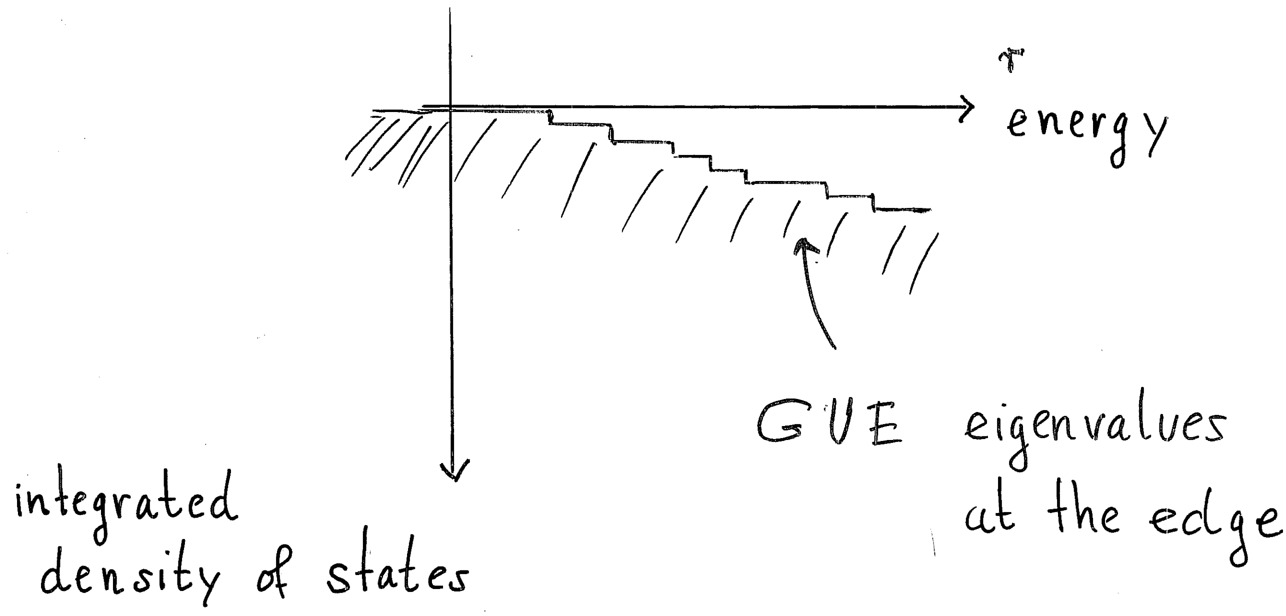
380°K

$- \tau^{3/2}$

Prokovsky/
Talapov
(1978)

g. 16. (a) Step-resolved image of (111) facet and vicinal surface of a Pb crystallite annealed at 380 K. (b) Line sca
om image of Fig. 16a showing sequence of monatomic steps, fit by Eq. (6).

Bonzel, Phys. Rep. 385 (2003)


energy
GUE eigenvalues
at the edge
integrated
density of states

large N

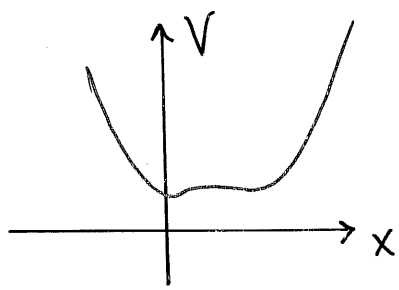
2. (Determinantal) line ensembles

A] matrix-valued diffusions

[multi-matrix model

$$dA(t) = -V'(A(t))\,dt + dB(t)$$

- $A(t)$, $B(t)$ are $N \times N$ complex matrices
- $V : \mathbb{R} \to \mathbb{R}$ confining



- $B(t) = B(t)^*$ GUE Brownian motion

$t \mapsto \langle f, B(t) f \rangle$ is Brownian motion

variance $\langle f, f \rangle^2 \, t$

$U^* B(t) U =_{dist} B(t)$ for all unitaries U

- $A(0) = A(0)^*$, then $A(t) = A(t)^*$ for all t

- invariant measure

$$\frac{1}{Z} e^{-2\operatorname{tr} V(A)} \, dA$$

of interest are:

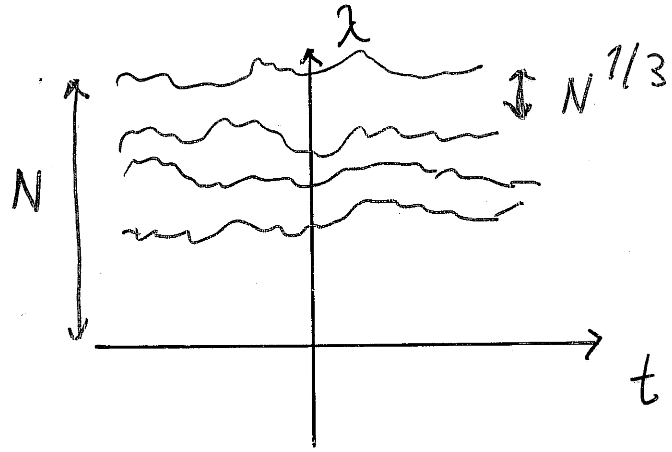
induced process of eigenvalues of $A(t)$

$$\lambda_1(t) < \dots < \lambda_N(t) \qquad \in \mathbb{R}$$

- example stationary OU process

$$V(x) = \frac{1}{2N} x^2$$

$$\lambda_{j+1} - \lambda_j = \mathcal{O}(1)$$



// top eigenvalue λ_N //

$$\lim_{N \to \infty} N^{-\frac{1}{3}} \left(\lambda_N (N^{2/3} t) - N \right) = \mathcal{A}_2(t)$$

Airy$_2$ process is stationary, has continuous sample paths

definition of $\mathcal{A}_2(t)$

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + x \qquad \text{on } L^2(\mathbb{R}, dx)$$

$\mathbb{P}$ projection onto $\{ H \le 0 \}$

Airy kernel

$$K(y,t; y',t') =$$

$$\left(e^{tH}\, \mathbb{P}\, e^{-t'H} - \mathbb{1}(t'>t)\, e^{-(t'-t)H} \right)(y,y')$$

//bounded//

$t_1 < \dots < t_m$

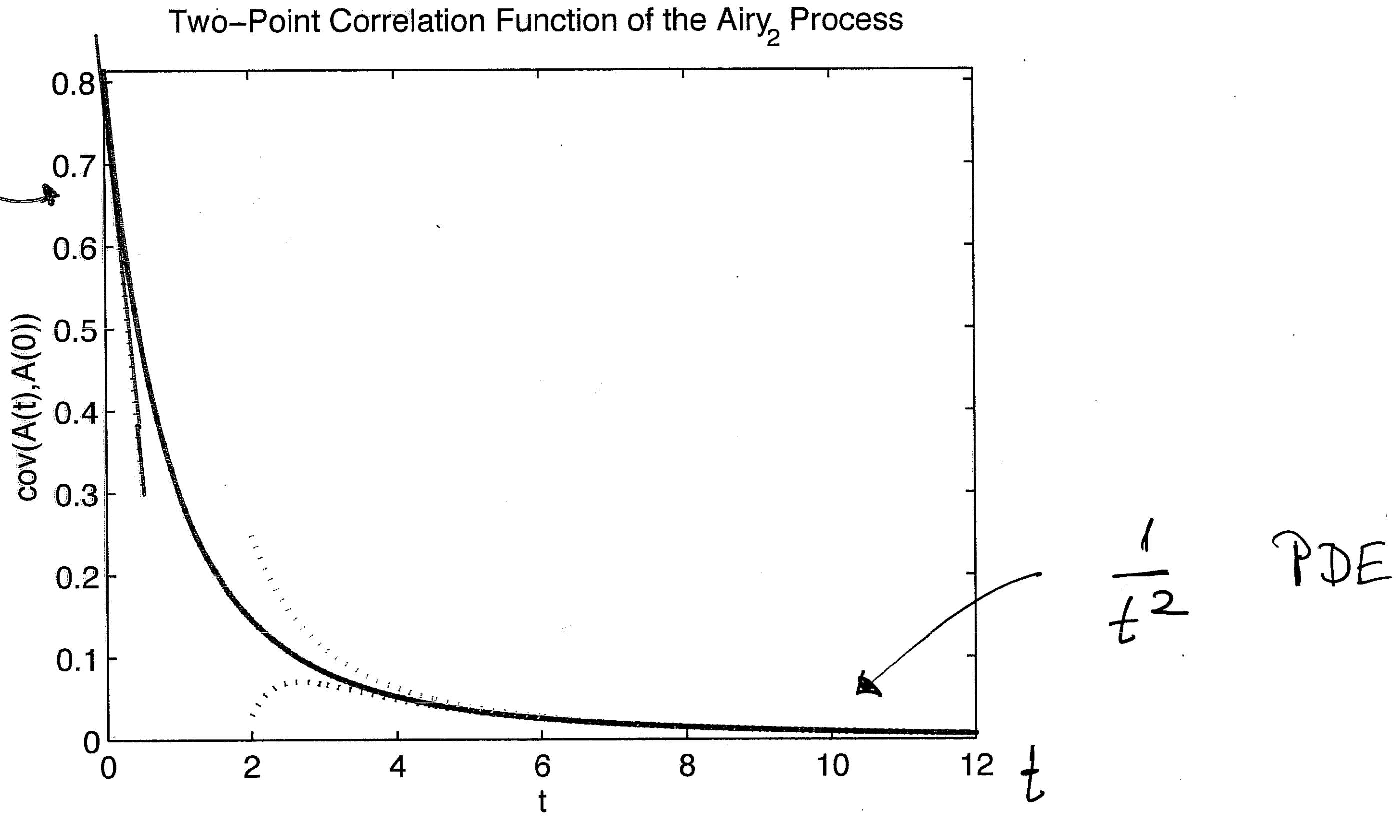
$$\mathbb{P}\left(\mathcal{A}_2(t_j) \le s_j,\ j=1,\dots,m \right) = \det\,(1 - R)$$

$$R_{ij}(y,y') = \mathbb{1}(y>s_i)\, K(y,t_i; y', t_j)\, \mathbb{1}(y'>s_j)$$

as operator on $\mathbb{C}^m \otimes L^2(\mathbb{R})$

// $m=1$ Tracy-Widom

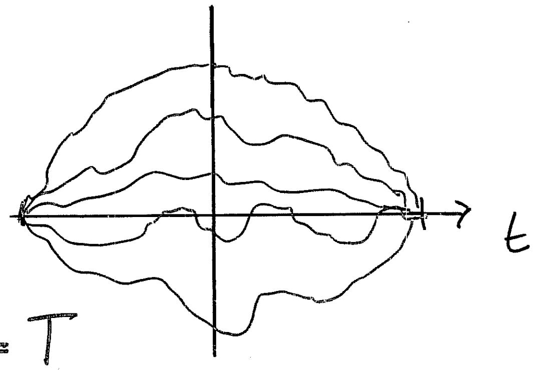
$$\langle \mathcal{A}i_2(0)\, \mathcal{A}_2(t) \rangle_T$$



Bornemann (2008)

• example Brownian bridge over $[-T, T]$
"water melon"

$$A(t) = B(T+t) - \frac{T+t}{2T} B(2T)$$



top eigenvalue λ_N^{BB}, $N = T$

$$\lim_{N \to \infty} N^{-1/3} \left(\lambda_N^{BB}(N^{2/3} t) - N \right) + t^2 = \mathcal{A}_2(t)$$

// lines with perfect repulsion // $\Rightarrow$ crystal

N independent Brownian bridges

$$x_j(t) = b_j(T+t) - \frac{T+t}{2T} b_j(2T)$$

$j = 1, \ldots, N$

• condition on nonintersection

$\rightsquigarrow$

$$\lambda_j^{BB}(t) =_{dist} x_j(t), \qquad |t| \le T$$

B] Growth processes

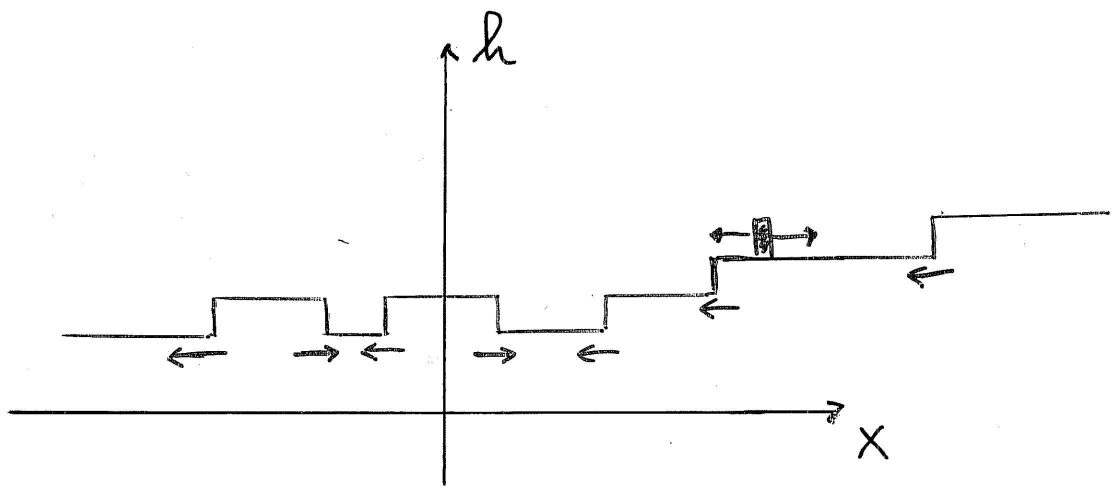
KPZ ?

discretized version

Poly Nuclear Growth model

height function $h(x,t)$, $x \in \mathbb{R}$, $t \geq 0$

$h \in \mathbb{Z}$ step size ± 1



- dynamics
  - deterministic: speed 1, coalescence
  - random: nucleation

    uniform space-time Poisson

line ensemble ?

HERE: droplet, nucleation only for $|x| < t$

top line $h = h_0$ PNG

extra lines $h_{-1}, h_{-2}, \ldots$

$$h_j(x,t) < h_{j+1}(x,t) \quad , \quad |x| < t$$

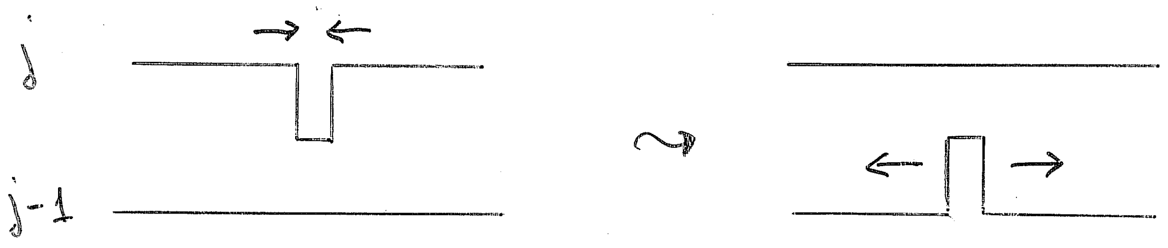
$$h_j(\pm t, t) = j$$

RULES:

random nucleation only for h_0

deterministic dynamics for all lines

- COPY coalescence at line j
  TO nucleation at line $j-1$



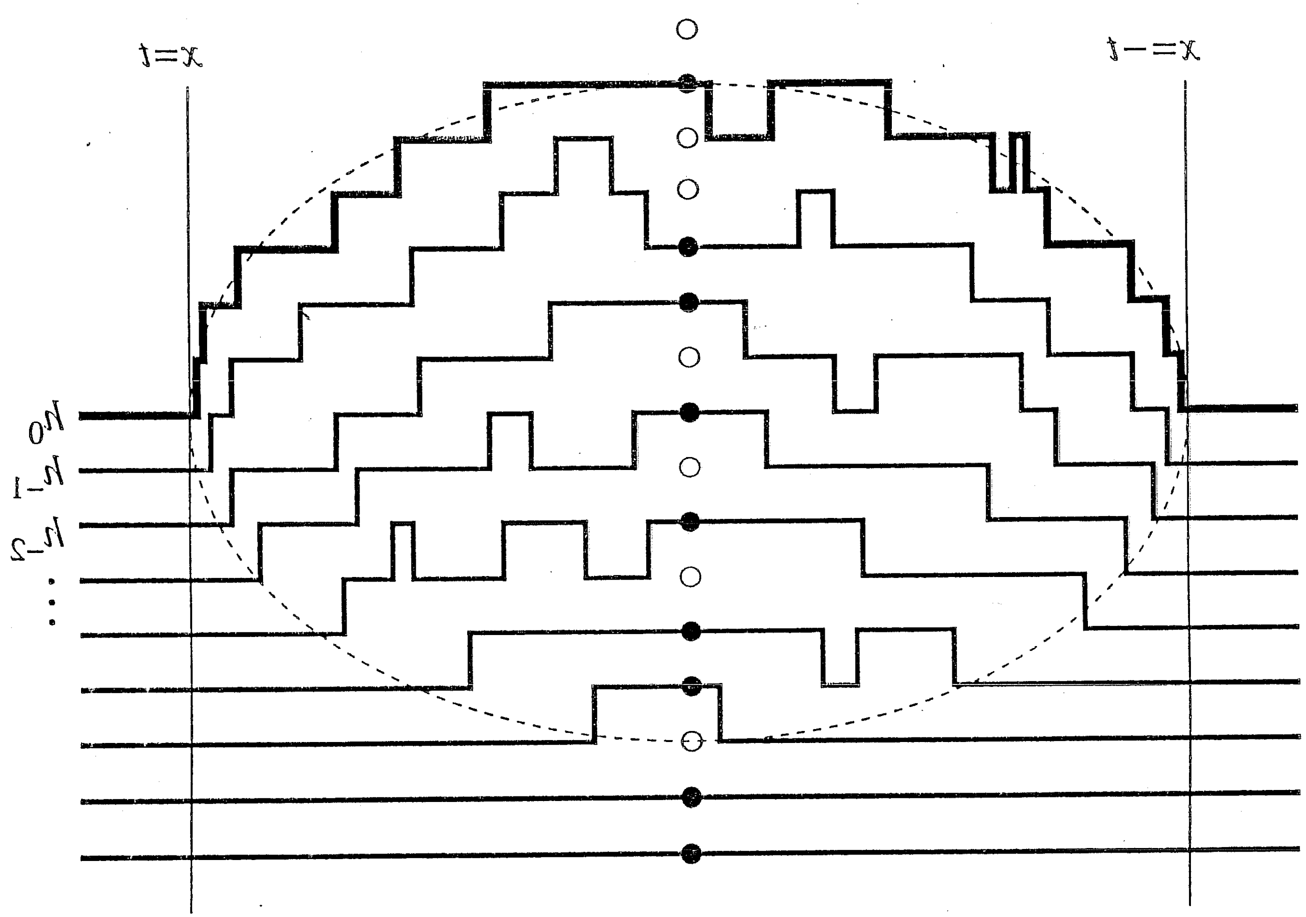


Figure 1: A snapshot of a multi-layer PNG configuration at time t. As a guide to the eye the asymptotic droplet shape is indicated.

Claim:

line ensemble has perfect repulsion

$w_j(x)$, $|x| \le t$, $j = 0, -1, -2, \dots$

independent, symmetric random walks

$w_j(\pm t) = j$

$\Rightarrow$ condition on nonintersection

$\rightsquigarrow$ $w_j(x) = h_j(x,t)$ $j = 0, -1, \dots$

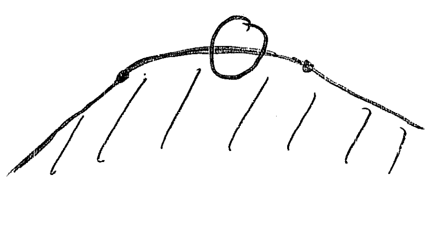
„growth model (PNG)

$$\lim_{t\to\infty} t^{-1/3}\left(h(t^{2/3}x, t) - 2t\right) + t^2 = \mathcal{A}_2(x)$$

expected to hold

whenever the macroscopic surface has

non-zero curvature

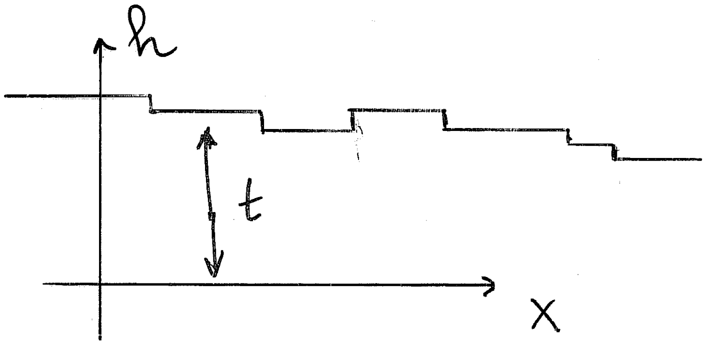


3. Flat initial conditions

PNG nucleation everywhere

$h(x, t=0) = 0$ (flat)

$x \mapsto h(x,t)$ is stationary



line ensemble (NOT determinantal)

$h_j(x,t)$, $j = 0, -1, -2, \ldots$ $h = h_0$

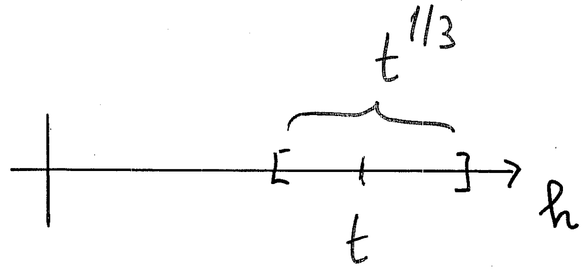
- $h_0(0,t) \cong 2t + t^{1/3} \underbrace{\xi_1}$

GOE Tracy-Widom distributed

Baik + Rains (2001)

- edge

$\{ h_j(0,t),\ j = 0, -1, \dots \}$



$t \to \infty$

top GOE eigenvalues Ferrari (2005)

natural conjecture

$x \mapsto h_0(x,t)$ large t

has statitistics of top line of GOE matrix-valued diffusion

NO!

Sasamoto, Borodin + Ferrari + Prähofer (2007) (2008)

$$\lim_{t \to \infty} t^{-1/3} \left(h(t^{2/3} x, t) - 2t \right) = \mathcal{A}_1(x)$$

same structure as $\mathcal{A}_2(x)$

extended Airy kernel is replaced by

$$B(y,y') = Ai(y+y') \,, \quad H = -\frac{d}{dy^2}$$

$$K_1(y,t;y',t')$$

$$= \left(-e^{-(t'-t)H} \mathbb{1}(t'>t) + e^{tH} B e^{-t'H}\right)(y,y')$$

covariance

$$\langle A_1(x) A_1(0) \rangle_T \simeq e^{-|x|}$$

GOE 2-matrix

real symmetric

$$\frac{1}{Z} \exp\left[-\frac{1}{N}\frac{1}{1-q^2} \operatorname{tr}\left(A_1^2 + A_2^2 - 2q A_1 A_2\right)\right] dA_1 dA_2$$

$$q = e^{-|x|/2N}$$

numerically covariance $\cong \dfrac{1}{x^2}$

flat $\neq$ flat

- microscopically flat $\quad h(x,0)=0$
- macroscopically flat

$h(0,0)=0$, $x \mapsto h(x,0)$ is random walk

$\frac{\partial}{\partial x} h(x,0)$ is stationary

for PNG

$$h(0,t) = 2t + t^{1/3} \underbrace{\xi_0}_{}$$

neither GOE, GUE

Conclusions

- line ensembles — RM / growth

identical edge statistics

third player:

directed last passage percolation

= directed polymer in random potential

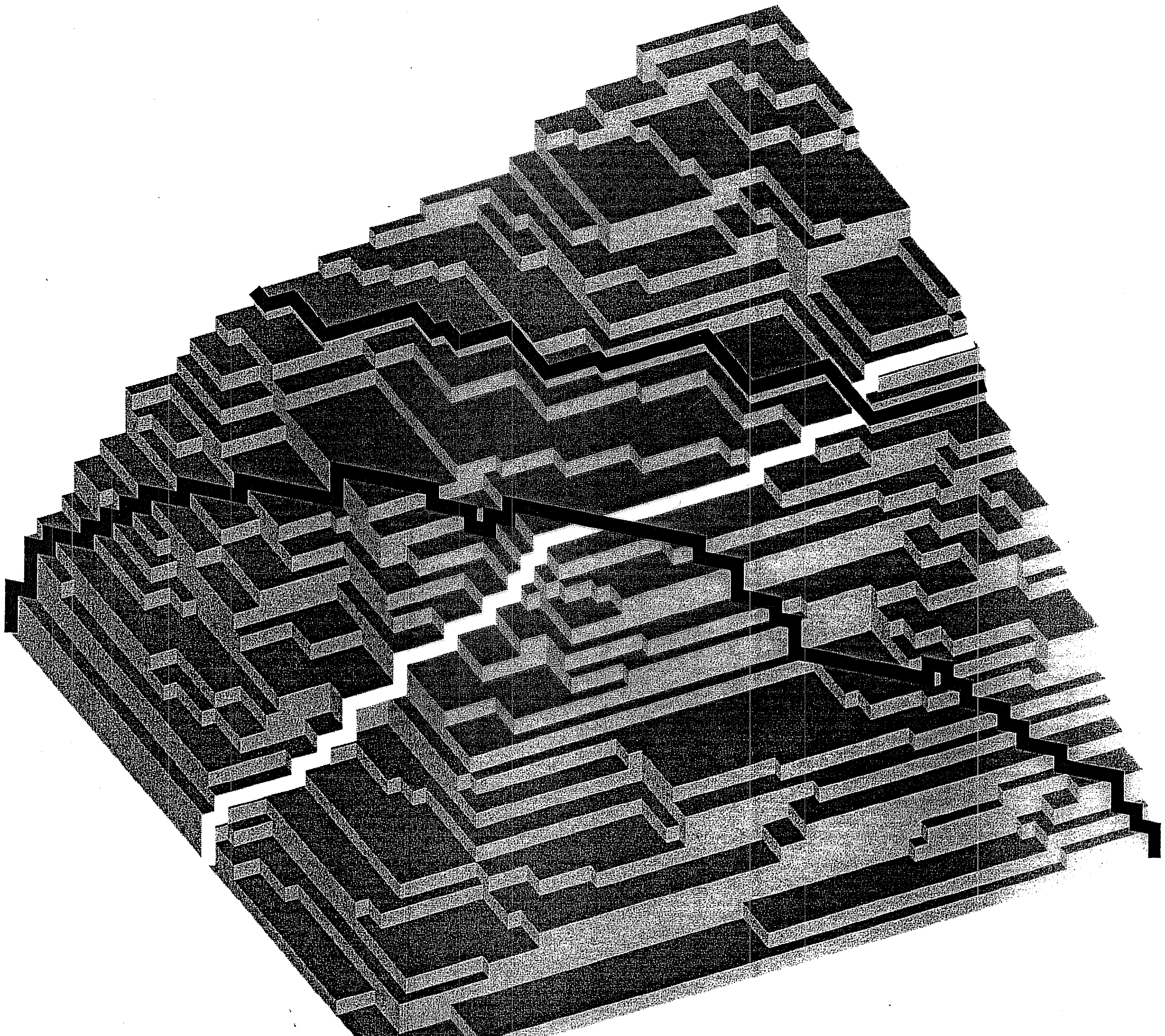
fourth player: 2D tiling

OPEN PROBLEM (for PNG droplet)

joint statistics of

$$\{ h(0,\tau t) , h(0,t) \}$$

for $t \to \infty$



energy mountain of directed polymer