The Real Ginibre Ensemble — An Integrable Structure

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$$d\mu(J_{ij}) = \prod_{ij=1}^{N} \left(\frac{dJ_{ij}}{\sqrt{2\pi}} e^{-J_{ij}^2/2} \right)$$

Ginibre 1965

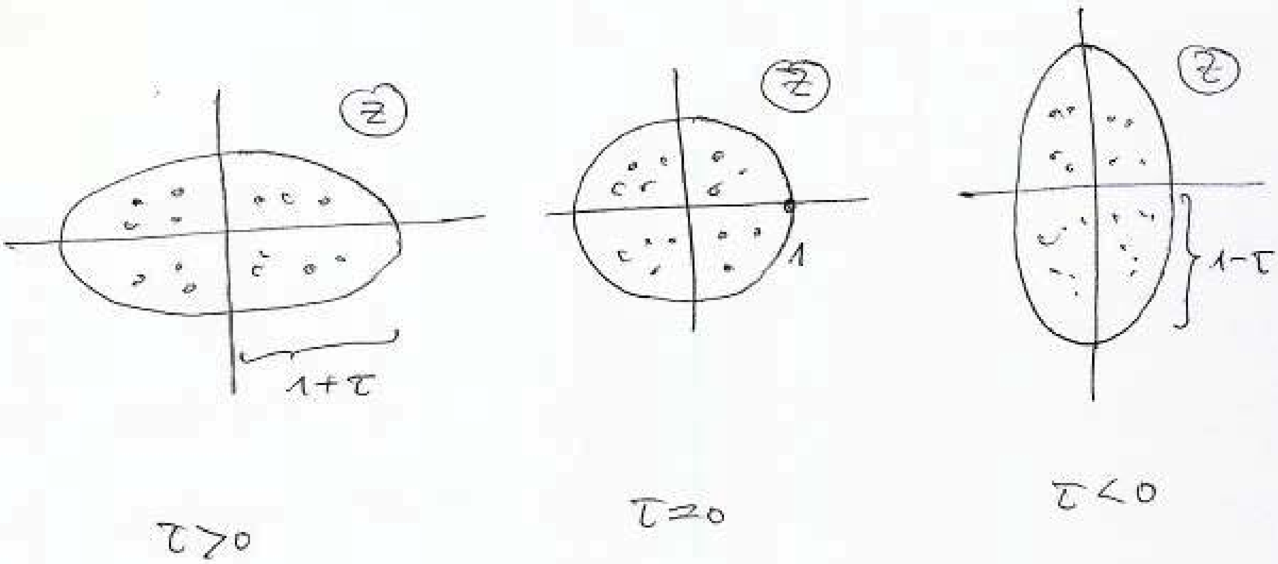
$$\det(J_{ij} - z\,\delta_{ij}) = 0 \quad \Rightarrow$$

$z = x$ real

or $z = x \pm iy$ complex conj.

Spectrum of large random asymmetric matrices

H.-J. S., Crisanti, Sompolinsky, Stein (1988)



$$\overline{J_{ij}^2} = \frac{1}{N}, \quad \overline{J_{ij} J_{ji}} = \frac{1}{N} \cdot \tau \qquad (i \neq j)$$

$R_1(z) = \text{const.}$ inside ellipse

$\sqrt{N}$ eigenvalues on real axis

Hermitian Ensembles $\quad H = H^{+}$

$$d\mu(H) = \mathcal{D}H \cdot \exp\left(-\tfrac{1}{2}\mathrm{Tr}\, H^2\right)$$

$$d\mu(E_1, E_2, \ldots, E_N) \propto dE_1 \ldots dE_N \prod_{i<j}(E_i - E_j)^\beta \prod_k e^{-E_k^2/2}$$

$$\text{with} \quad \beta = \begin{cases} 1 & \\ 2 & \text{for} \\ 4 & \end{cases} \quad \begin{matrix} \text{GOE} \\ \text{GUE} \\ \text{GSE} \end{matrix}$$

eigenvalue correlations: Pfaffians

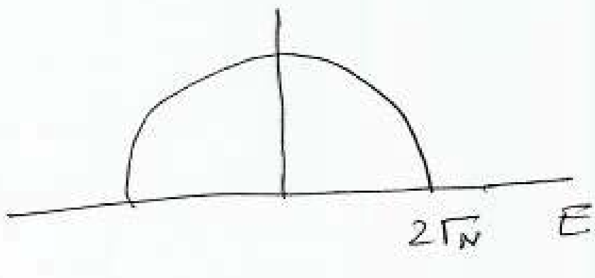
GUE : $\beta = 2$

$$R_n(E_1, \cdots E_n) = \det\left(K_N(E_i, E_j)\right) \qquad i,j = 1, 2, \cdots n$$

$$K_N(E_1, E_2) = \frac{1}{\sqrt{2\pi}} e^{-(E_1^2 + E_2^2)/4} \sum_{n=0}^{N-1} p_n(E_1)\, p_n(E_2)$$

(Hermite)

$$R_1(E) \simeq \frac{1}{\sqrt{2\pi}} \sqrt{4N - E^2}$$



Wigner semicircle Law

Complex Ginibre Ensemble

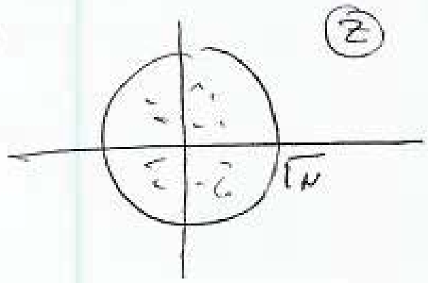
$$d\mu(J) = \prod_{ij=1}^{N} \left(\frac{d^2 J_{ij}}{\pi} e^{-|J_{ij}|^2} \right)$$

$$d\mu(z_1, z_2, \dots z_N) \propto e^{-\sum_i |z_i|^2} \prod_{i<j} |z_i - z_j|^2 \, d^2 z_1 \cdots d^2 z_N$$

$$R_n(z_1, \dots z_n) = \det\left(K_N(z_i, z_j) \right) \quad , \; i,j = 1, 2 \dots, n$$

$$K_N(z_1, z_2) = \frac{e^{-(|z_1|^2 + |z_2|^2)/2}}{\pi} \sum_{n=0}^{N-1} \frac{(z_1 \bar{z}_2)^n}{n!}$$

$$R_1(z) \simeq \frac{1}{\pi} \, \theta(\sqrt{N} - |z|)$$



$$R_2(z_1, z_2) \simeq \frac{1}{\pi} \left(1 - e^{-|z_1 - z_2|^2} \right) \quad \text{for } N \to \infty$$

Real Ginibre Ensemble

joint probability density

$$d\mu(z_1, \cdots, z_N) = C_N \, dz_1 \cdots dz_N \prod_{i<j}(z_i - z_j) \prod_k f(z_k)$$

different cases: $2n$ compl. conj. $0 \leq n \leq \frac{N}{2}$

ordered such that $d\mu \geq 0$.

Lehmann, S. (1991)

$$f(z)^2 = e^{-x^2} \cdot e^{y^2} \cdot \mathrm{erfc}(|y|\sqrt{2}) \quad \text{with } z = x + iy$$

density of complex eigenvalues

$$R_1^c(z) \propto |y| \, f(z)^2 \cdot \sum_{n=0}^{N-2} \frac{|z|^{2n}}{n!}$$

Edelman (1997)

Normalisation constant:

$$\frac{1}{C_N} = \mathrm{Pfaff}\,(A_{kl}) \qquad k,l = 1,2,\dots N$$

$$= \int dx_1 \dots dx_N \, \exp\left(-\tfrac{1}{2} \sum_{kl} x_k A_{kl} x_l\right)$$

$$x_k x_l = -x_l x_k$$

$$A_{kl} = \int d^2 z_1 \int d^2 z_2 \; z_1^{k-1} z_2^{l-1} \mathcal{F}(z_1, z_2) \quad = - A_{lk}$$

$$\mathcal{F}(z_1, z_2) = - \mathcal{F}(z_2, z_1)$$

$$= f(z_1) f(z_2) \left(2i\, \delta^2(z_1, \bar{z}_2)\, \mathrm{sgn}\, y_2 + \delta(y_1)\delta(y_2)\, \mathrm{sgn}(x_2 - x_1) \right)$$

Kernel

$$K_N(z_1, z_2) = \sum_{k,l=1}^{N} z_1^{k-1} z_2^{l-1} A^{-1}_{kl}$$

Use

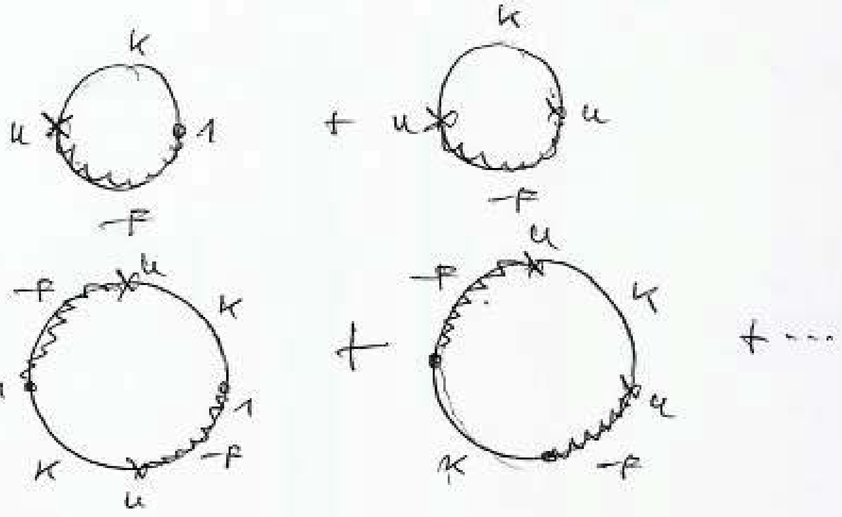
$$\prod_{i<j} (z_i - z_j) = \int dx_1 \dots dx_N \prod_{n=1}^{N} \left(\sum_k x_k z_n^{k-1} \right)$$

Generating function for correlations

$$Z[u] = \mathrm{Pfaff}\left(\tilde{A}_{kl}(u)\right)$$

with $f(z) \to f(z)\,(1+u(z))$

$\ln(Z/Z_0) =$



= sum of simple fermion loops

n-point densities

$$R_n(z_1, \dots z_n) = \frac{\delta^n}{\delta u(z_1) \dots \delta u(z_n)} Z[u] \Big|_{u=0}$$

= Pfaffians.

$$R_n(z_1, \cdots, z_n) = \left| \mathrm{Pfaff} \begin{bmatrix} K & -G \\ G^T & -W \end{bmatrix} \right|$$

Akemann Kanzieper, Forrester Nagao
Sinclair Borodin, H.-J. S. Wieczorek

with

$$K_{ij} = K_N(z_i, z_j) \qquad i,j = 1, 2, \cdots n$$

$$G_{ij} = -\int d^2z\, K_N(z_i, z)\, \mathcal{F}(z, z_j)$$

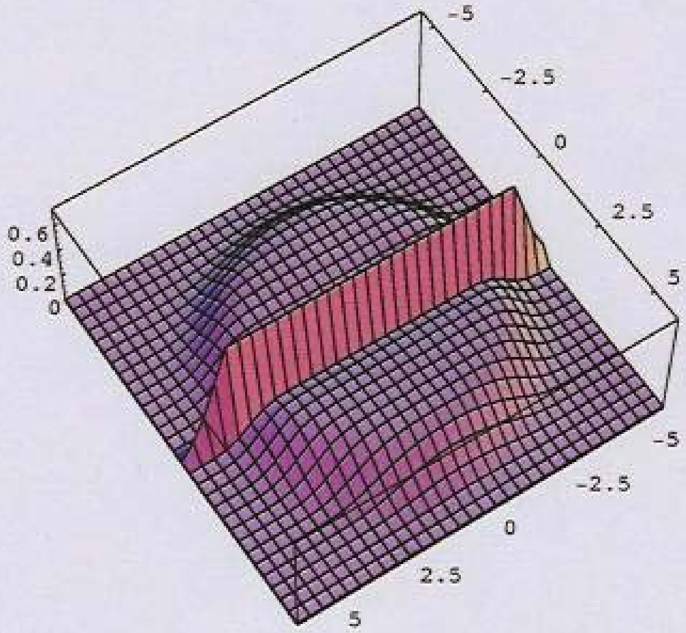
$$W_{ij} = -\mathcal{F}(z_i, z_j) + \int d^2z \int d^2z'\, \mathcal{F}(z_i, z)\, K_N(z, z')\, \mathcal{F}(z', z_j)$$

$$R_1(z_1) = \int d^2z\, K_N(z_1, z)\, \mathcal{F}(z, z_1)$$

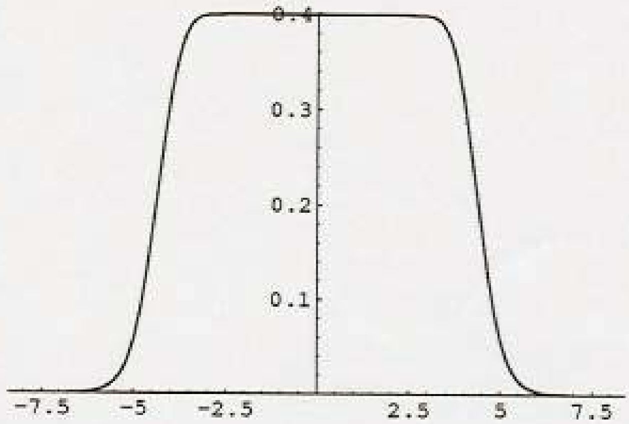
$$= \underbrace{R_1^C(z_1)}_{\text{compl.}} + \delta(y_1)\, \underbrace{R_1^R(x_1)}_{\text{real}}$$

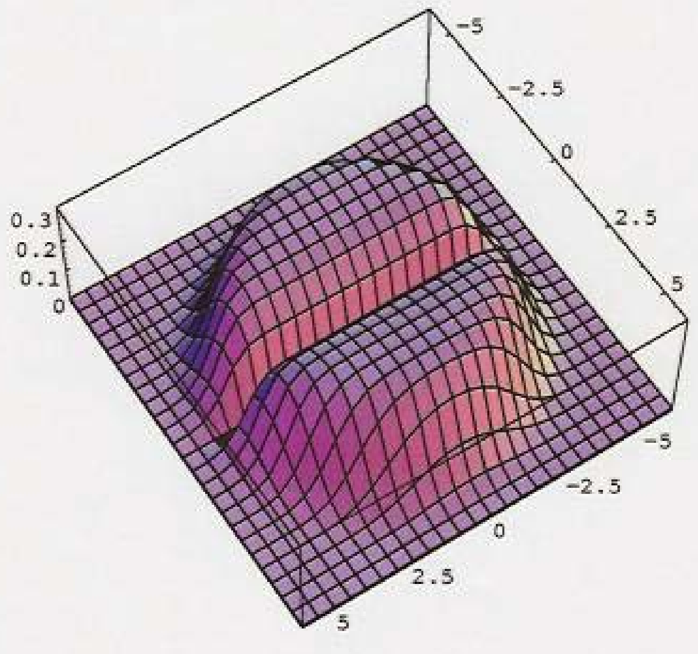
pictures

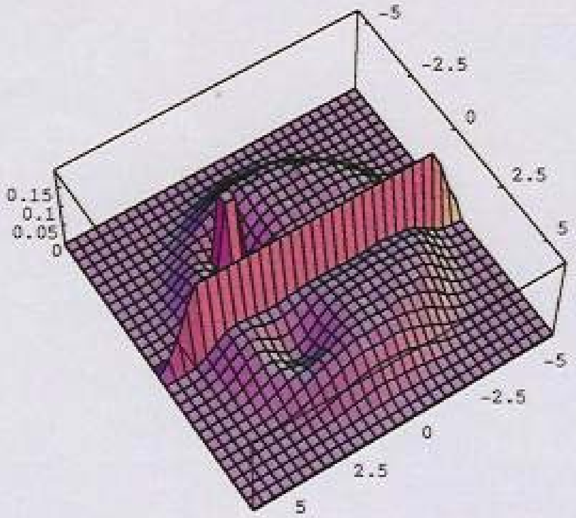
10a



10.b.







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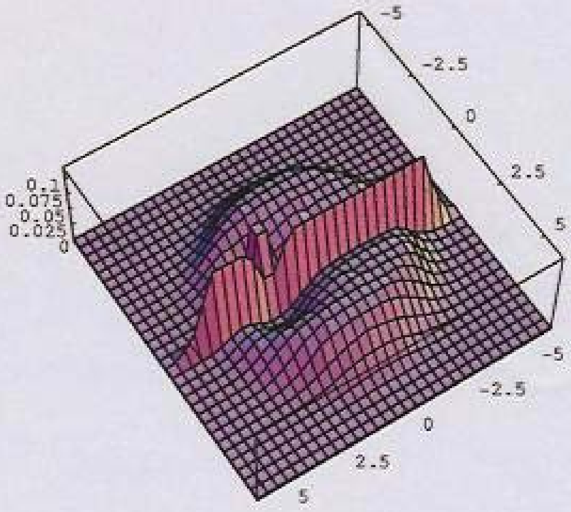

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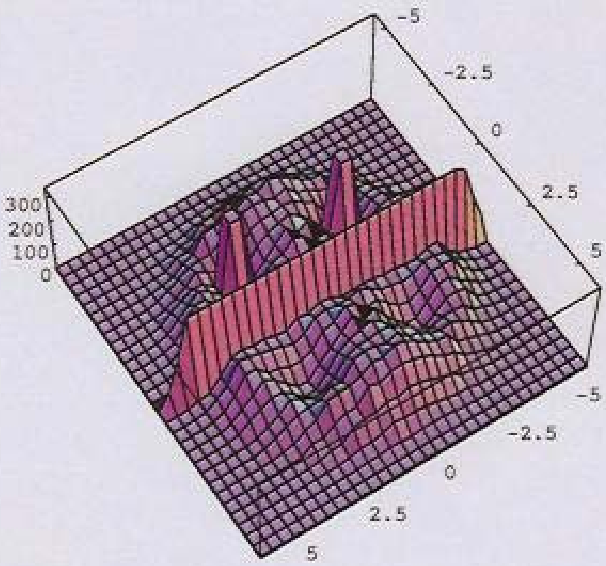

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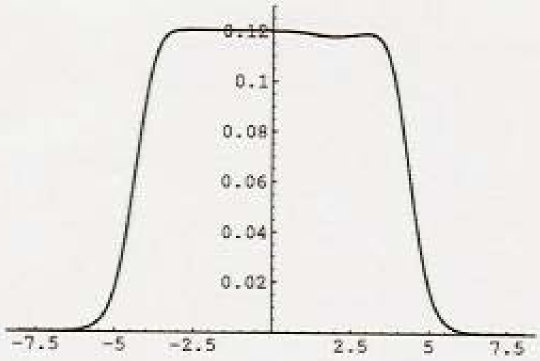
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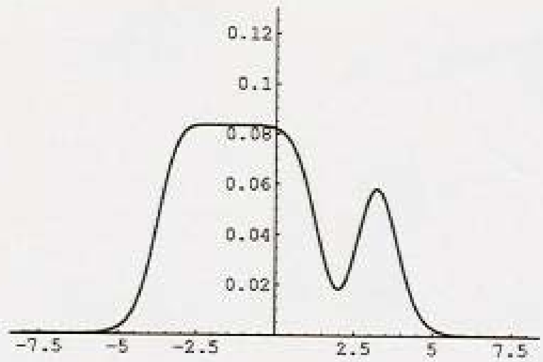
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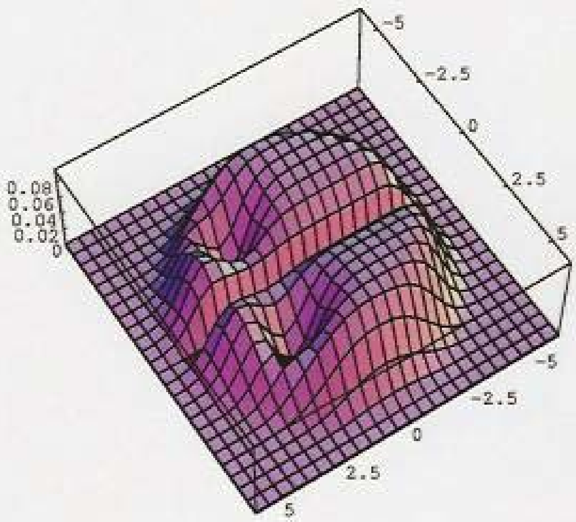
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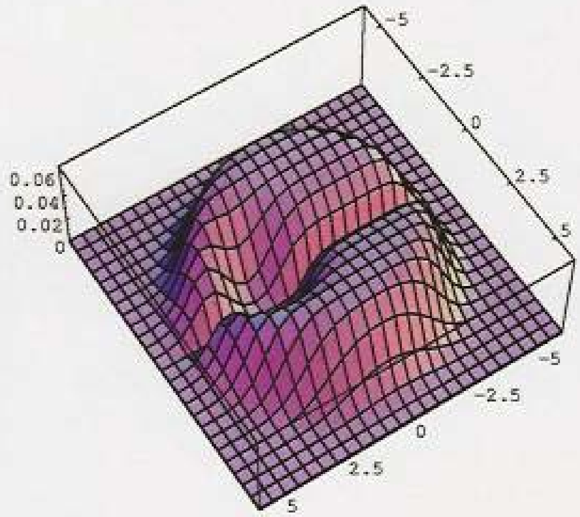
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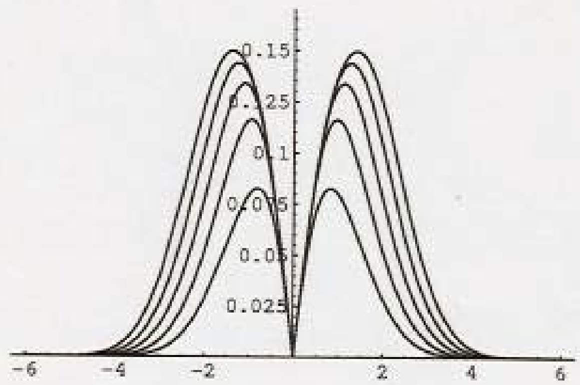
0.15
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-2
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6

16.a

Relation to characteristic polynomials

$$K_N(z_1, z_2) = \sum_{k,l=1}^{N} z_1^{k-1} z_2^{l-1} A^{-1}_{kl}$$

$$= \frac{(z_1 - z_2)}{2\sqrt{2\pi}} \sum_{n=0}^{N-2} \frac{(z_1 z_2)^n}{n!}$$

determines everything

$$R_1^c(z) \propto f(z)^2 \, K_N(z, \bar{z}) \qquad \text{(Edelman)}$$

$\Rightarrow$

$$K_{N+2}(z, \bar{z}) \propto (z - \bar{z}) \langle |\det(z - J)|^2 \rangle_N$$

$$\propto (z - \bar{z}) \left\langle \int D\eta \int D\xi \, \exp(-\eta^*(z - J)\eta - \xi^*(\bar{z} - J^T)\xi) \right\rangle_N$$

$$\propto (z - \bar{z}) \sum_{n=0}^{N} \frac{|z|^{2n}}{n!}$$

Akemann, Philips, S. 09

$\Rightarrow$

$$\langle \det(z - J) \det(\bar{z} - J^T) \rangle_N$$

determines everything

Hidden Structure

$$A^{-1}_{k\ell} = \frac{1}{2\sqrt{2\pi}} \begin{pmatrix} 0 & -\frac{1}{0!} & & \bigcirc \\ \frac{1}{0!} & 0 & -\frac{1}{1!} & \\ & \frac{1}{1!} & 0 & \ddots \\ \bigcirc & & \ddots & \end{pmatrix}$$

$$= a_k^{-1} \; \varepsilon_{k\ell} \; a_\ell^{-1} \quad , \quad a_k = 2^{k/2}\,\Gamma(k/2)$$

with duplication formula

$$2\sqrt{2\pi}\;\Gamma(N-1) = 2^{(N-1)/2}\,\Gamma((N-1)/2)\cdot 2^{N/2}\,\Gamma(N/2)$$

$$\varepsilon_{k\ell} = \begin{pmatrix} 0 & -1 & & & \bigcirc \\ 1 & 0 & -1 & & \\ & 1 & 0 & -1 & \\ & & 1 & 0 & \ddots \\ \bigcirc & & & \ddots & \end{pmatrix}$$

$$\overset{-1}{\varepsilon}_{kl} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \cdots \\ -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & \cdots \\ -1 & 0 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ -1 & 0 & -1 & 0 & -1 & 0 & \cdots \\ \vdots & & \vdots & & \vdots & & \ddots \end{pmatrix}$$

$$A_{kl} = a_k \, \overset{-1}{\varepsilon}_{kl} \, a_l$$

$$= \int d^2 z_1 \int d^2 z_2 \; \mathcal{F}(z_1, z_2) \; z_1^{k-1} z_2^{l-1}$$

for any power, $k-1$, $l-1$

$\mathrm{Pfaff}(\varepsilon^{-1}) = 1.$

Schur functions averages

Schur functions

$$\sigma_\lambda(z_i) = \frac{\det\left(z_m^{\,N-n+\lambda_n}\right)}{\det\left(z_m^{\,N-n}\right)}$$

partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0.$

$\Rightarrow$

$\langle \sigma_\lambda(z_i) \rangle_N \;\propto$

$$\mathrm{Pfaff} \int d^2 z_1 \int d^2 z_2 \;\mathcal{F}(z_1, z_2)\, z_1^{N-n+\lambda_n}\; z_2^{N-m+\lambda_m}$$

$$= \prod_{k=1}^{N} a_{N-k+\lambda_k+1}\, \mathrm{Pfaff}\left(\overleftarrow{\varepsilon}^{\,1}_{N-k+\lambda_k,\; N-l+\lambda_l} \right)$$

Result:

$$\langle \sigma_\lambda(z_i) \rangle_N = \langle \sigma_\lambda(J) \rangle_N = \prod_{n=1}^{N} a_{N-n+\lambda_n+1} \Big/ \prod_{n=1}^{N} a_n$$

$$= 2^{|\lambda|/2} \prod_{n=1}^{N} \frac{\Gamma((N-n+\lambda_n+1)/2)}{\Gamma((N-n+1)/2)}$$

for all λ_n even

$= 0$ otherwise

$|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_N$

also valid for odd N.

H.-J. S. Khoruzhenko arXiv
0903.5071

Conclusions

joint probability density

Pfaffian correlations

skew symmetric kernel

relation to characteristic polynomials

Shur function averages

relation to τ- function, theory of integrability ?