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Gap Probabilities in Non-Hermitian Random Matrix Theory

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collaboration with

M J Phillips, L Shifrin

arXiv:0901.0897

Outline

- **Motivation:**

  - Why non-Hermitian
  - Why gaps: applications

- **RMTs:** Ginibre & chiral complex $\beta = 2, 4$

- **Method:** (skew) orthogonal polynomials on $\mathbb{C}$

- **New results for:**

  - generic non-Hermiticity (+ expansion & surmise)
  - max non-Hermiticity: explicit gap & universal relation $\beta = 2 \leftrightarrow 4$

- **Open Questions**

Motivation: Why Non-Hermitian RMT?

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Examples for non-Hermitan operators:

- **Hamiltonian** $H \neq H^\dagger$ e.g. of open systems:
  scattering & absorption
  (NOT Pseudo Hermitian $H \neq H^\dagger$ with real spectrum)

- **Dirac operator** $D(\mu) \neq D(\mu)^\dagger$ with μ chemical potential:
  Fermion # not conserved = grand canonical

- related Q's:
  - what if zeros of $\zeta(s)$ were complex - change in correlations?
  - root of real polynomials

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Real eigenvalues:

- **spacing distribution** = probability that interval empty

  - $\exists$ simple formula = **Wigner surmise**
  - universal

- **individual eigenvalues** $\leftrightarrow$ probability all levels $\leq s$ $(\geq s)$

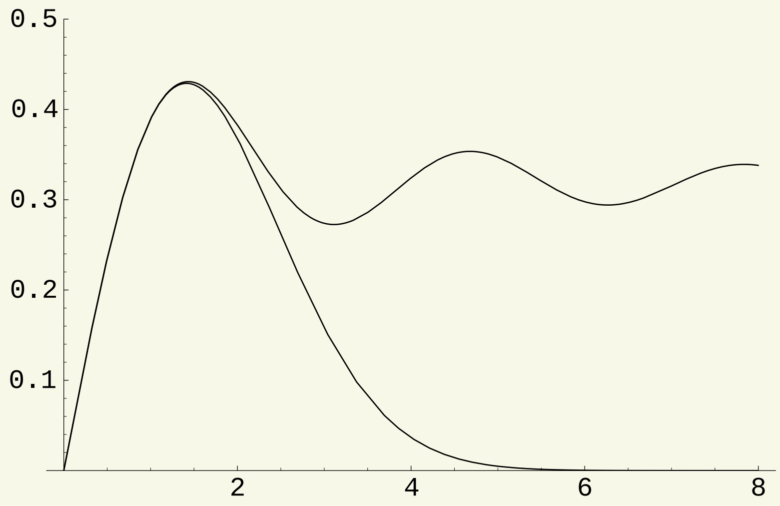
  - largest eigenvalue: **Airy** law / **Tracy-Widom** distribution
  - smallest eigenvalue on $\mathbb{R}_+$: **Bessel** law

  - universal & many applications
  - more difficult than spectral densities

Q: What happens when levels becomes **complex**?

$\rightarrow$ problem remains integrable

Example Bessel Law: Real vs Complex



Bessel-density
vs 1st ev on $\mathbb{R}$

$$p(x) \sim x \exp[-\frac{1}{4}x^2]$$

[Forrester; Damgaard, Nishigaki, Wettig; Guhr, Wettig, Wilke]

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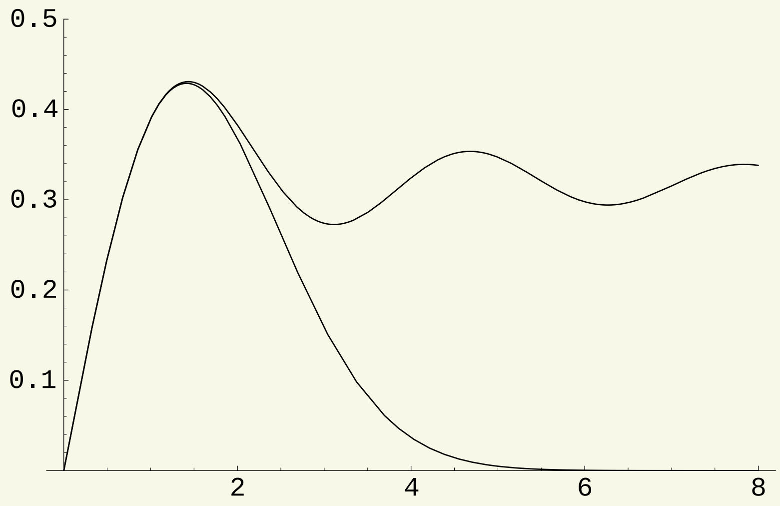

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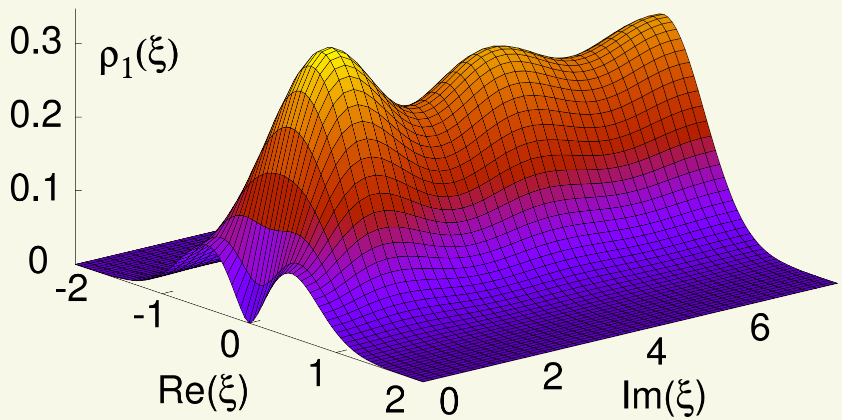

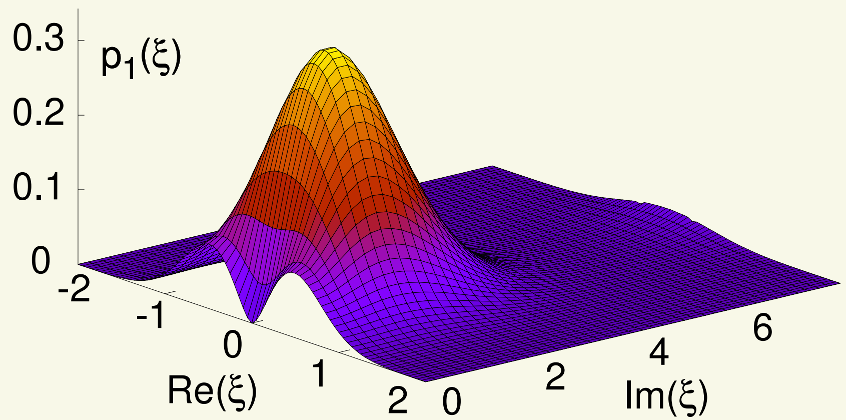

- known density on $\mathbb{C}$ [Splittorff, Verbaarschot] vs **approximate 1st ev** [Bloch, Shifrin, Wettig, G.A.]

QCD Lattice Data: Real vs Complex

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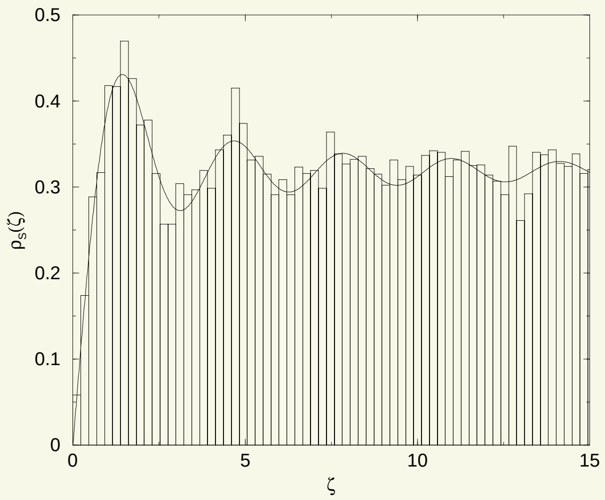

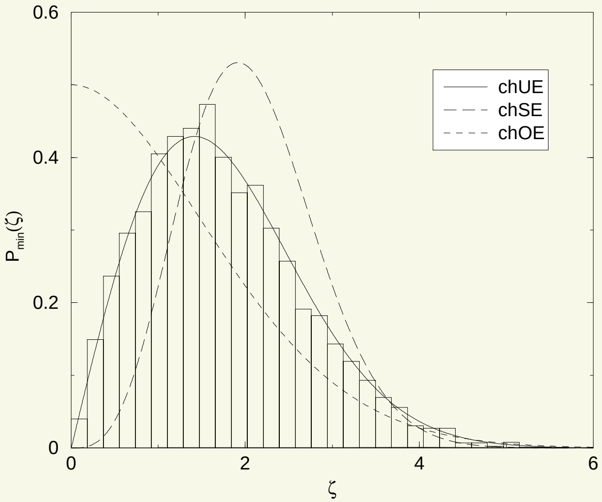

[Damgaard et al.]

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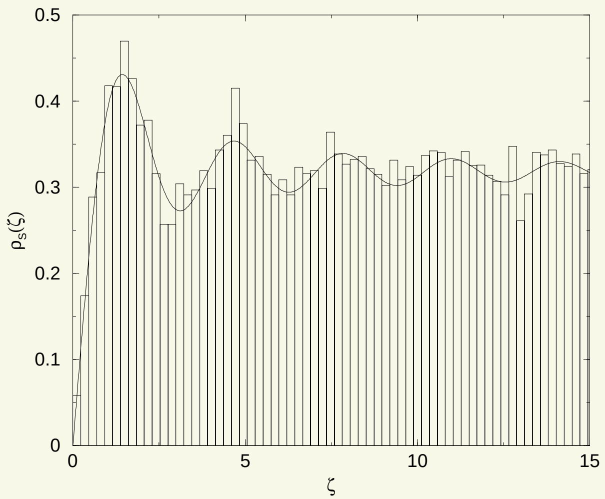

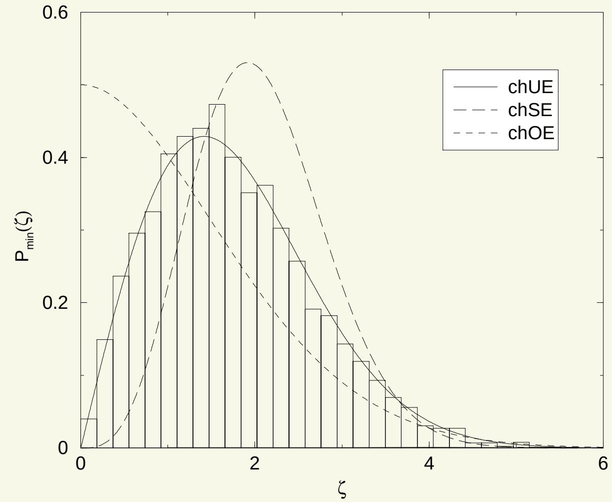

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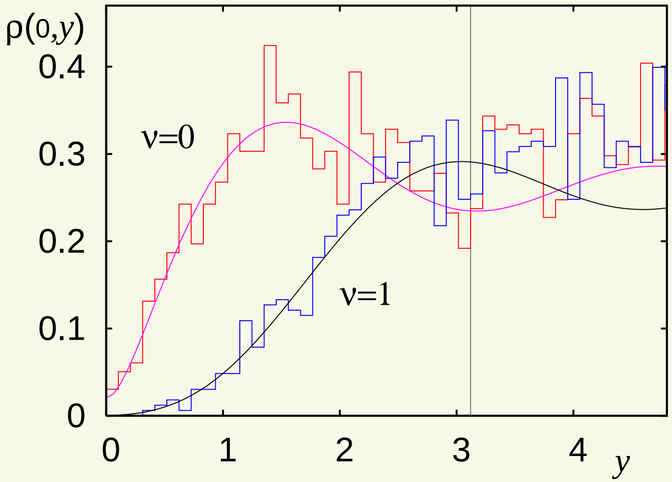

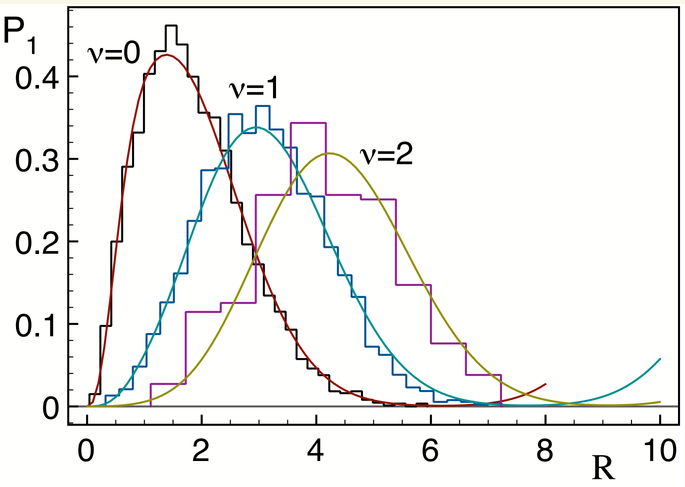

[Bloch, Wettig

+ Shifrin, G.A.]

Which Non-Hermitian RMTs

- Ginibre ensembles: $J = H + iv(\tau)A$ [Fyodorov, Khoruzhenko, Sommers; Kanzieper]

$$\boxed{\mathcal{Z} = \int dJ \exp\left[-\frac{N}{1-\tau}\mathrm{Tr}(JJ^\dagger + \tau\Re\mathrm{e}J^2)\right]}$$

- chiral complex ensembles: W, V non-Hermitan [Osborn; G.A.]

$$\boxed{\mathcal{Z} = \int dW dV \det{}^{N_F}\begin{pmatrix} 0 & iW + \mu V \\ iW^\dagger + \mu V^\dagger & 0 \end{pmatrix} e^{-N\mathrm{Tr}(WW^\dagger + VV^\dagger)}}$$

- matrices $N \times N(+\nu)$, with complex/ real quaternion elements

  τ, μ non-Hermiticity parameters

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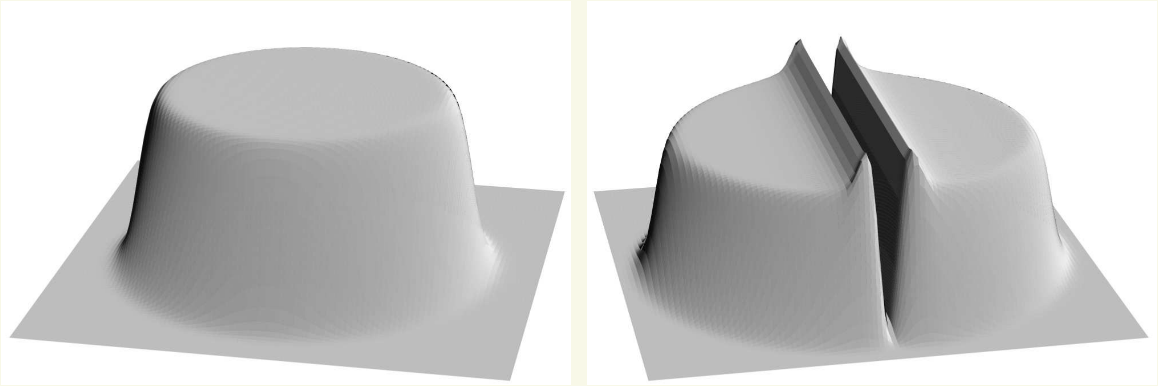
  τ, μ non-Hermiticity parameters

- **weak non-Hermitity** [FKS]: **scale** $N(1-\tau)$ or $N\mu^2$

  - Physics applications: scattering $M \ll N$, **QCD** & SU(2) + μ

  - **interpolates** real $\leftrightarrow$ strongly complex correlations

Example Ginibre: flat $\beta = 2$ vs. extra repulsion for $\beta = 4$



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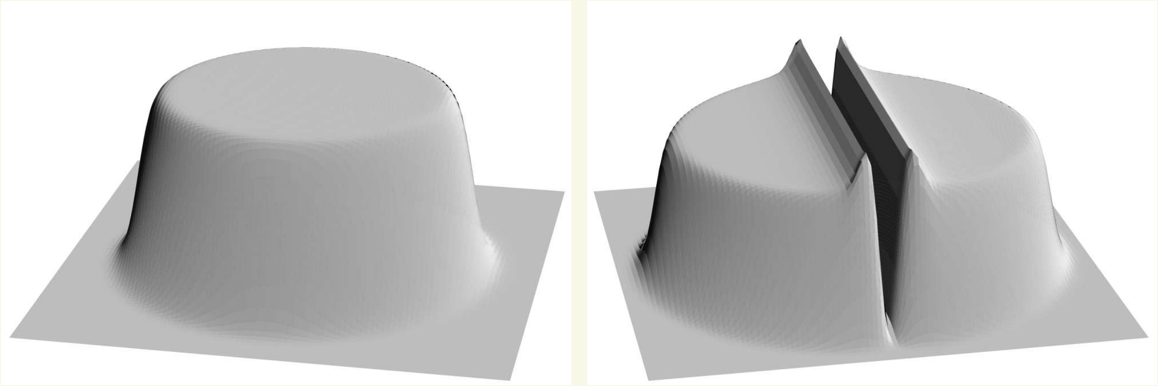

- chiral complex models: **more repulsion** at origin & $\Re$e
- real Ginibre / chiral complex: exact real / + imaginary ev

  [Kanzieper, G.A.; Sommers+ Wieczorek; Forrester, Nagao; Borodin, Sinclair; Phillips, Sommers, G.A.]
- total # of classes: 10 + 33 [Zirnbauer, Altland; Bernard, LeClair]

Method: (skew) Orthogonal Polynomials

- **complex ev:** Jacobian (chiral $z \to z^2$)

$$\mathcal{Z} \sim \int_{\mathbb{C}} \prod_i^N d^2 z_i \, w(z_i) \begin{cases} |\Delta_N(z)|^2 & \beta = 2 \\ \prod_j^N (z_j - z_j^*) \Delta_{2N}(z, z^*) & \beta = 4 \quad \neq \text{ normal} \end{cases}$$

- **weight $w(z)$:**

- Ginibre: Gauß

- chiral: $\sim \exp[b(z^2 + z^{*\,2})] K_\nu(a|z|^2)$ **non-Gauß** for Gauß RMT!

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- **integrability: all densities** ρ_k

$$\rho_k(z_1, \ldots, z_k) \sim \int d^2 z_{k+1} \ldots d^2 z_N \mathcal{P}_{JPDF} = \det_{k \times k}[\mathcal{K}_N(z_i, z_j^*)]$$

for β=2 (β=4 Pfaffian)

- **Kernel** $\mathcal{K}_N(z, z^*)$ of **complex (skew) Orthogonal Polynomials**

  $P_k(z)$ classical Hermite / Laguerre polynomials

Gap From Fredholm Eigenvalues

- def circular gap $\boxed{E(\mathbf{r}) \sim \prod_j^N \int_{\mathbf{r}}^{\infty} dr_j r_j \int_0^{2\pi} d\varphi_j \mathcal{P}_{JPDF}}$
- def 1st eigenvalue $\boxed{p(\mathbf{r}) = -\partial_{\mathbf{r}}\, E(\mathbf{r})}$

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Results: Fredholm

$\underline{\beta = 2:}$ $$E(r) = \mathbf{det}_{k,j} \left[\int_r^{\infty} dr r \int_0^{2\pi} d\varphi\, w(z) P_k(z) P_j(z^*) \right] = \prod_{k=0}^{N-1} (1 - \lambda_k^{(2)})$$

$\underline{\beta = 4:}$ $$E(r) = \mathbf{Pf}_{k,j} \left[\int_r^{\infty} dr r \int_0^{2\pi} d\varphi\, w(z)(z - z^*)[q_k(z) q_j(z^*) - (k \leftrightarrow j)] \right]$$
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$$= \prod_{k=0}^{N-1} (1 - \lambda_k^{(4)})$$

- **<u>generic non-Hermiticity & "Gauß":</u>**
  – matrix elements recursively: Bessel-I and -K BUT λ_j difficult
- k-th gap: **generating functional** $E^{(\beta)}(r;\xi) = \prod_k (1 - \xi \lambda_k(r))$

Maximal Non-Hermiticity = Rotational Invariance

- **general weight $w = w(|z|)$:** $P_k(z) \sim z^k$

  $\boxed{\int_0^{2\pi} d\varphi\, z^k z^{*\,j} \sim \delta_{kj}}$ diagonalises Fredholm

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$\Rightarrow$ **universal relation on** $\mathbb{C}$**:** $\beta = 2 \leftrightarrow 4 \;\; \forall\, w(|z|)$

$$\boxed{E^{(\beta=\mathbf{4})}(r) = \prod_K^N (1 - \lambda_{2K+1}^{(\beta=\mathbf{2})})}$$ **skip even**

(for Gauß Ginibre see also [Mehta])

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- **different relation on** $\mathbb{R}$**:** even plus odd

$$\boxed{E_{\mathbb{R}}^{(\beta=\mathbf{4})}(r) = \prod_K (1 - \lambda_{2K}^{(\beta=\mathbf{2})}) \; + \; \prod_K (1 - \lambda_{2K+1}^{(\beta=\mathbf{2})})}$$

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Fredholm eigenvalues explicitly known:

- **chiral ensembles:**

$$1 - \lambda_k^{(\beta=2)}(r) = aK_\nu + b(\hat{I}_{\nu+1}K_{\nu+2} + \hat{I}_{\nu+2}K_{\nu+1})$$

- linear combination of Bessel-K and **incomplete Bessel-$\hat{I}$**

- solution of the eigenvalue equation:

$$\lambda\Psi(z) = \int_0^x dt t^2 \int_0^{2\pi} d\theta |z| K_\nu(t^2)^{\frac{1}{2}} K_\nu(|z|)^{\frac{1}{2}} I_\nu(zte^{-i\theta})\Psi(te^{i\theta})$$

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- **Ginibre ensembles:** [Grobe, Haake, Sommers]

$$1 - \lambda_k^{(\beta=2)}(r) = \Gamma(k+1, r)$$

- incomplete Gamma function (exp times **incomplete exp**)

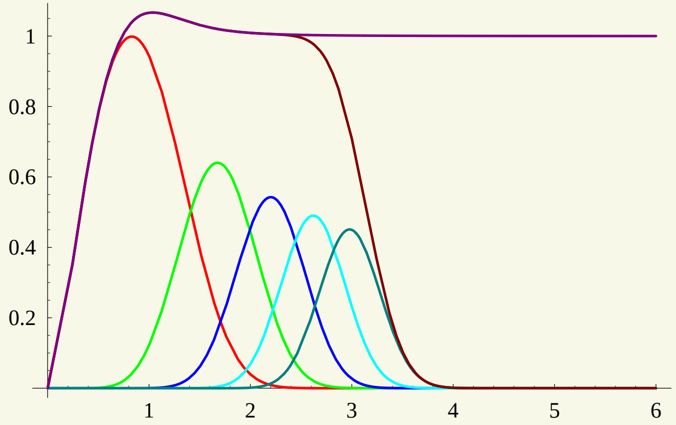
$$\lambda\Psi(z) = \int_0^x dtt^2 \int_0^{2\pi} d\theta e^{-\frac{1}{2}(t^2+|z|^2)+zte^{-i\theta}} \Psi(te^{i\theta})$$

Gaps vs Densities: Expand

all individual ev p_k known $\Leftrightarrow$ all densities ρ_k known

- **example density:** $\boxed{\rho(z) = \sum_k p_k(z)}$

  radial density



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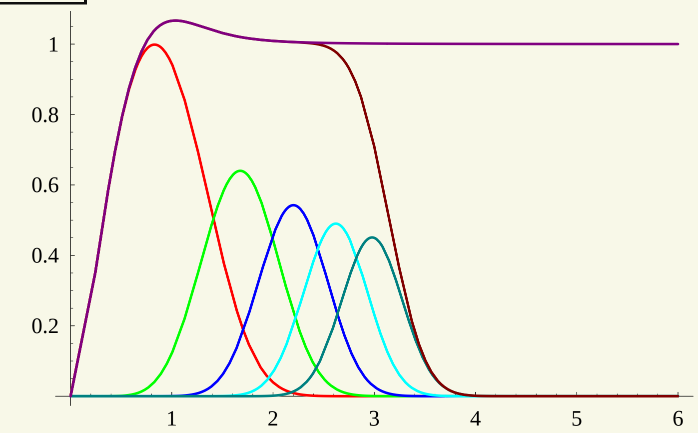
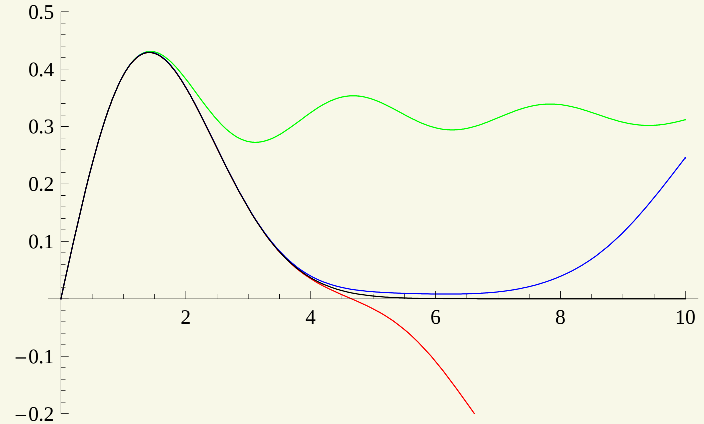


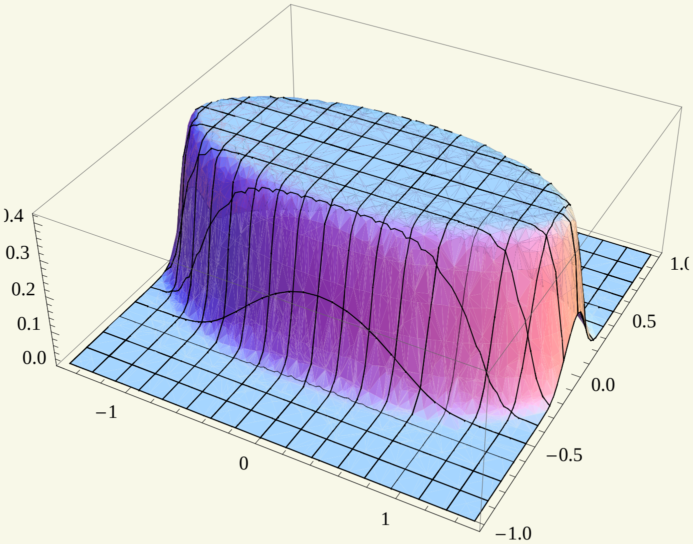
- **example gap:** $\boxed{E(r) = \sum_k \int \cdots \int \rho_k} = 1 - \int_0^r \rho(z) + \int \int_0^r \rho(z,u) + \ldots$

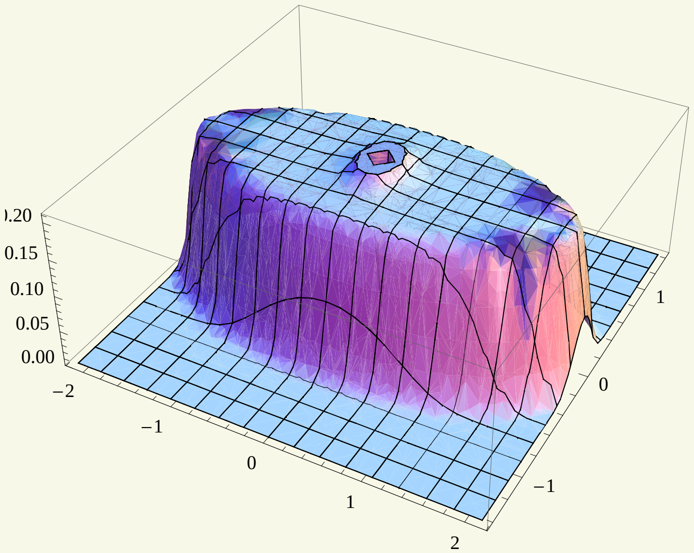
  convergent expansion: $\mathbb{R}$ [Damgaard, G.A.] and $\mathbb{C}$ [ABSW]



Spacing Distribution: Small Radius

- Ginibre: translation invariance: 1 ev at origin $\Rightarrow$ spacing = gap [GHS]
- chiral complex: put exact 1 zero ev $\nu = 1$ at origin ($\neq$ bulk !)





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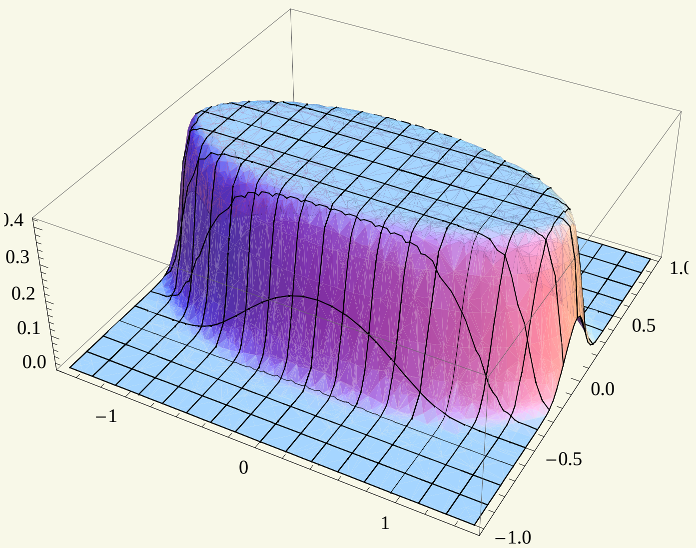
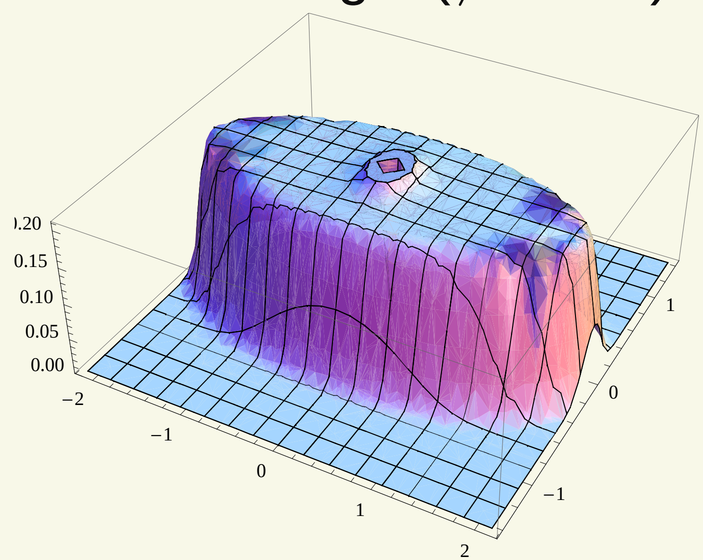




- $\boxed{P(s) \sim s^3}$ universal for $\underline{\beta = 2}$ [Oas], **here chiral**

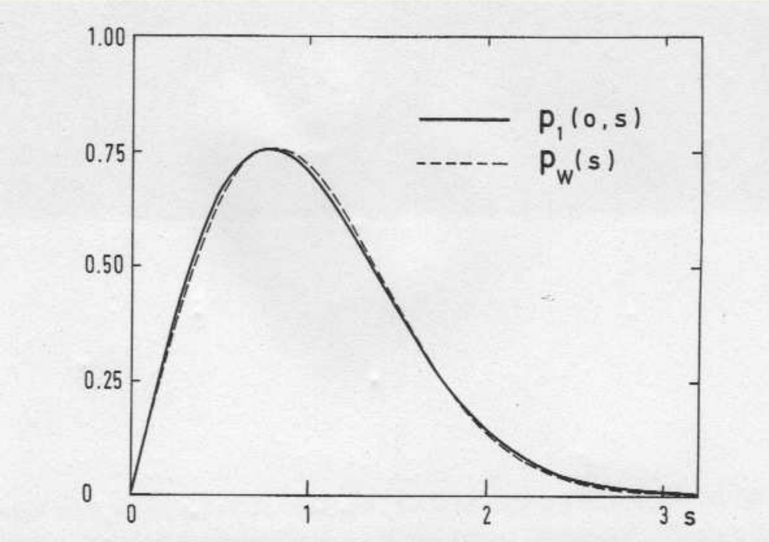
  Q: stronger than on $\mathbb{R}$ $\underline{P(s) \sim s^\beta}$? No: 3 = 2 + 1 from measure

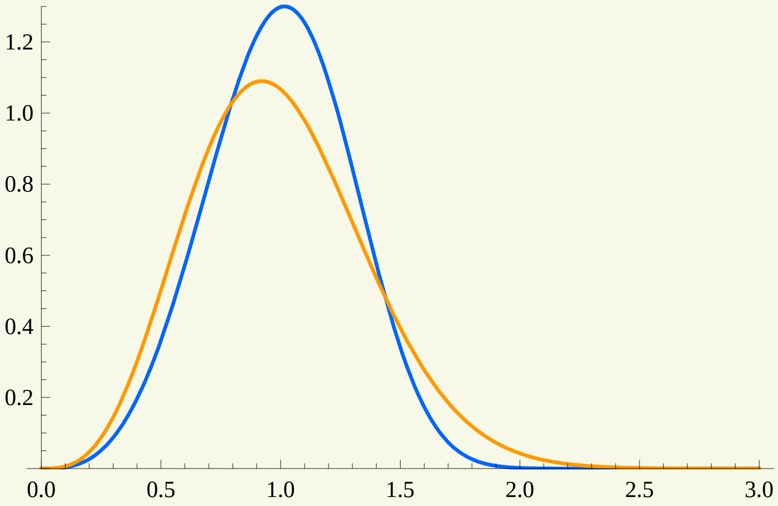
- $\boxed{P(s) \sim s^7}$ for chiral $\underline{\beta = 4}$ at $2\nu = 1$ (origin !)

  - extra powers can also be understood from measure

A Wigner Surmise for Complex?

known: **Wigner-Dyson:** YES, **Ginibre:** NO [Grobe,Haake,Sommers]





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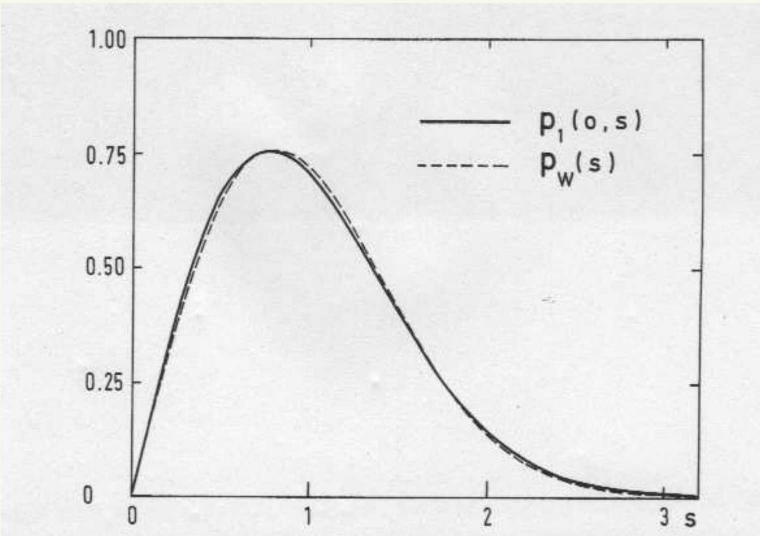
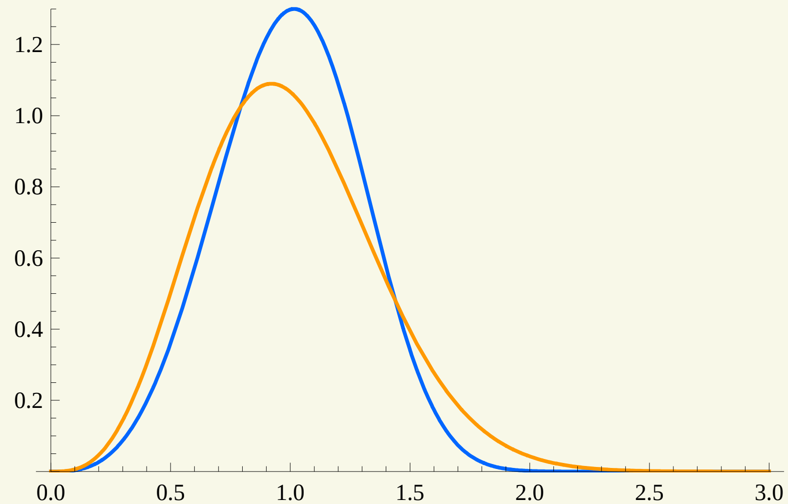
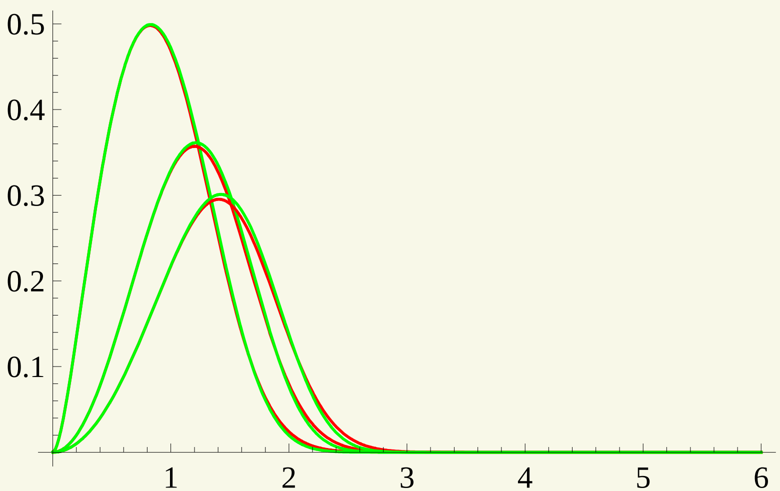
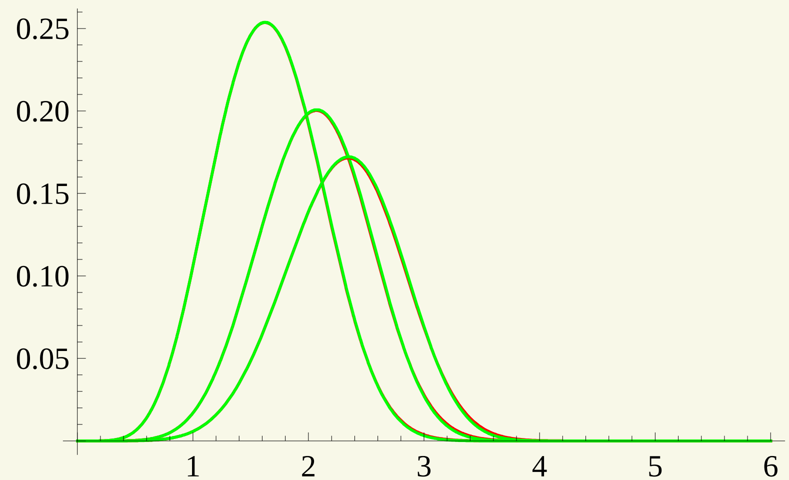




chiral complex: YES ($\beta = 2$ left, $\beta = 4$ right; $\nu = 0, 1, 2$)





Large Radius Asymtotics

- $\lim_{x\to\infty} E_{chiral}(x) \equiv e^{-P(x)} = \prod_k (1-\lambda_L)$ **eff index $L = 2k + \nu + \frac{1}{2}$**
- $\boxed{P(x) = \frac{1}{4}x^2 + x\frac{1}{2}\log(x) + xC}$ [Forrester], where $x = Nr^2$ and $N \to \infty$
- relation

	Ginibre	chiral
$\beta = 2$	$C = \frac{1}{2}\log(2\pi)$	$\frac{1}{2}C - \frac{\nu}{2}$
$\beta = 4$	$\frac{1}{2}C - \frac{1}{4}$	$\frac{1}{4}C - \frac{1}{4}(2\nu+1)$

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Ensemble	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5	$\frac{11}{2}$	6	$\frac{13}{2}$
Ginibre $\beta = 2$	•		•		•		•		•		•		•	
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Ginibre $\beta = 2$	•		•		•		•		•		•		•	
Ginibre $\beta = 4$			•				•				•			
Chiral $\beta = 2$ ($\nu = 0$)		•				•				•				•
Chiral $\beta = 4$ ($\nu = 0$)						•								•

Open Problems

- Large-N limits
- More explicit result for weak non-Hermiticity gaps?
- Gaps in real Ginibre / chiral complex
- Universality in chiral complex ensembles!?