Painlevé Transcendents and Quantum Transport

Eugene Kanzieper

Department of Applied Mathematics
H.I.T. - Holon Institute of Technology
Israel

ISF Research Workshop on Random Matrices and Integrability, Yad Hashmona, March 26, 2009

Painlevé Transcendents and Quantum Transport

Eugene Kanzieper

in collaboration with

Vladimir Al. Osipov

Universität Duisburg-Essen
Germany

ISF Research Workshop on Random Matrices and Integrability, Yad Hashmona, March 26, 2009

- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model
  - 1D impenetrable Bose gas
  - Growth models
- New! Painlevé transcendents in quantum transport problems: Announcement of the main result
- Landauer conductance and its cumulants: Known results
- Integrable theory of quantum transport in chaotic cavities
  - Landauer conductance
  - Noise power (thermal-to-shot-noise crossover)
- Conclusions / Open problems

Painlevé transcendents and their appearance in physics

$$y'' = F(x, y, y')$$

Rational function of its arguments — **E. Picard** (1889)

All **movable singularities** are restricted to **poles**
(no movable branch points)

50 → 44

(a) **linear 2nd order DEs**
(b) **Weierstrass DE**
(c) **Riccati DE**

$$(y')^2 = 4y^3 - g_2 y - g_3$$

$$y' = a(x)\, y^2 + b(x)\, y + c(x)$$

[6 **Painlevé equations P_I – P_{VI}**
nonlinear special functions]

P. Painlevé (1900,1902)
B. Gambier (1905)
R. Fuchs (1910)

Painlevé transcend

$$y'' = F(x, y, y')$$

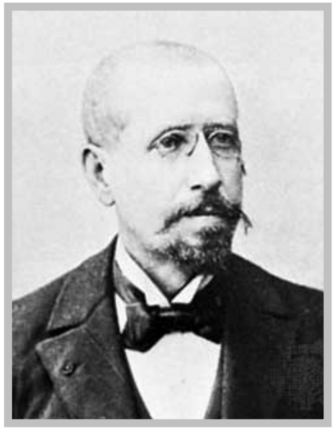
50 → 44

(c) **Riccati DE**

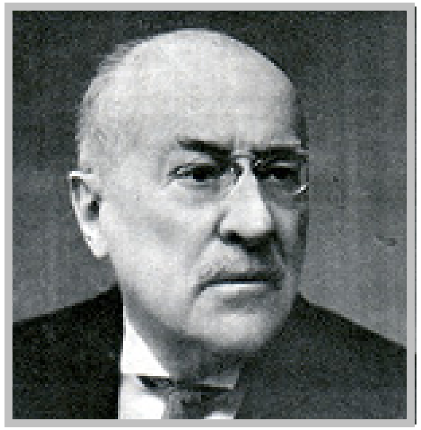
$$y' = a(x)\, y^2 + b(x)\, y + c(x)$$

[6 **Painlevé equations P_I – P_{VI}** nonlinear special functions]

P. Painlevé (1900,1902)
B. Gambier (1905)
R. Fuchs (1910)



Gaston Darboux
(1842 - 1917)



Emile Picard
(1856 - 1941)

Paul Painlevé
(1863 - 1933)

- **Painlevé transcendents and their appearance in physics**

σP$_{\mathrm{III}}$

$$\left(t\sigma''_{\mathrm{III}}\right)^2 - \nu_1\nu_2\left(\sigma'_{\mathrm{III}}\right)^2 + \sigma'_{\mathrm{III}}(4\sigma'_{\mathrm{III}} - 1)(\sigma_{\mathrm{III}} - t\sigma'_{\mathrm{III}}) - \frac{1}{4^3}\left(\nu_1 - \nu_2\right)^2 = 0$$

σP$_{\mathrm{V}}$

$$\left(t\sigma''_{\mathrm{V}}\right)^2 - \left[\sigma_{\mathrm{V}} - t\sigma'_{\mathrm{V}} + 2(\sigma'_{\mathrm{V}})^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma'_{\mathrm{V}}\right]^2 + 4(\nu_0 + \sigma'_{\mathrm{V}})(\nu_1 + \sigma'_{\mathrm{V}})(\nu_2 + \sigma'_{\mathrm{V}})(\nu_3 + \sigma'_{\mathrm{V}}) = 0$$

[6 **Painlevé equations P$_{\mathrm{I}}$ – P$_{\mathrm{VI}}$**
nonlinear special functions]

P. Painlevé (1900,1902)
B. Gambier (1905)
R. Fuchs (1910)

P. Clarkson, Painleve equations – nonlinear special functions, J. Comp. Appl. Math. 153, 127 (2003)

▸ Painlevé transcendents and their appearance in physics

σP_{III}

$$\left(t\sigma_{III}''\right)^2 - \nu_1\nu_2\left(\sigma_{III}'\right)^2 + \sigma_{III}'(4\sigma_{III}' - 1)(\sigma_{III} - t\sigma_{III}') - \frac{1}{4^3}(\nu_1 - \nu_2)^2 = 0$$

σP_V

$$\left(t\sigma_V''\right)^2 - \left[\sigma_V - t\sigma_V' + 2(\sigma_V')^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma_V'\right]^2 + 4(\nu_0 + \sigma_V')(\nu_1 + \sigma_V')(\nu_2 + \sigma_V')(\nu_3 + \sigma_V') = 0$$

Hamiltonian system — Bäcklund transformations

[6 Painlevé equations P_I – P_{VI} nonlinear special functions]

fascinating properties

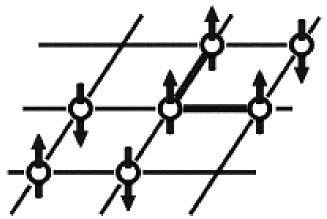
P. Clarkson, Painleve equations – nonlinear special functions, J. Comp. Appl. Math. 153, 127 (2003)

Painlevé transcendents and their appearance in physics

- **2D Ising model**

$$H_{\rm int}^{(2D)} = -J \sum_{j,k} \left(\sigma_{j,k}\, \sigma_{j,k+1} + \sigma_{j,k}\, \sigma_{j+1,k} \right)$$

R. Peierls (1936), H. Kramers & G. Wannier (1941)
L. Onsager (1944), C. N. Yang (1955)

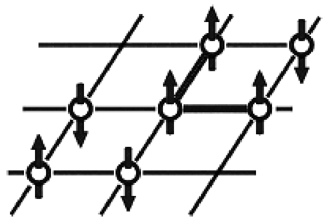


Existence of the 2[nd] order phase transition, Critical temperature, Spontaneous magnetisation

Painlevé transcendents and their appearance in physics

- **2D Ising model**

$$H^{(2\mathrm{D})}_{\mathrm{int}} = -J \sum_{j,k} \left(\boldsymbol{\sigma}_{j,k}\, \boldsymbol{\sigma}_{j,k+1} + \boldsymbol{\sigma}_{j,k}\, \boldsymbol{\sigma}_{j+1,k} \right)$$



Existence of the 2nd order phase transition, Critical temperature, Spontaneous magnetisation

T. Wu, B. McCoy, C. Tracy, and E. Barouch (1976)

$$\langle \boldsymbol{\sigma}_{00} \boldsymbol{\sigma}_{MN} \rangle \Big|_{\substack{T \to T_c^{\pm} \\ R=(M^2+N^2)^{1/2} \to \infty}} = \boldsymbol{F}^{\pm} \left([\sigma_{\mathrm{III}}];\, r = \frac{R}{\xi(T)} \right)$$

$$\xi(T) = \frac{(T_c/4J)}{|1 - T/T_c|}$$

σP_{III}

$$\tanh(J/T_c) = \sqrt{2} - 1$$

Painlevé transcendents and their appearance in physics

- 2D Isin

$$H_{\rm int}^{(2D)} =$$

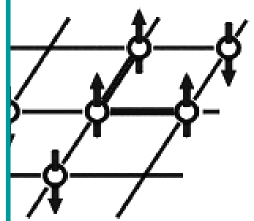
R. Peierls
T. Wu, B. ... L.

$$\langle \sigma_{00}\sigma_{M}$$

$$\tanh(J/T_c) = \sqrt{2} - 1$$

2007 Wiener Prize

The committee also recognizes the earlier work of Craig Tracy with Wu, McCoy, and Barouch, in which Painlevé functions appeared for the first time in exactly solvable statistical mechanical models. In addition, the committee recognizes the seminal contributions of Harold Widom to the asymptotic analysis of Toeplitz determinants and their various operator theoretic generalizations.



...e 2nd order phase
...tical temperature,
...magnetisation

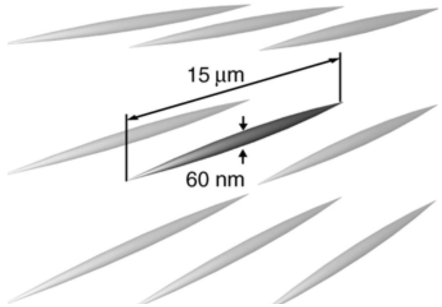
$$\ldots) = \frac{(T_c/4J)}{|1 - T/T_c|}$$

σP$_{III}$

Painlevé transcendents and their appearance in physics

- **Impenetrable Bose gas** $g \to \infty$

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + g \sum_{i<j} \delta(z_i - z_j)$$



© T. Esslinger ETH Zürich

FIG. 1. The geometry and size of trapped 1D gases in a two-dimensional optical lattice. The spacing between the 1D tubes in the horizontal and vertical direction is 413 nm.

M. Girardeau (1960)
A. Lenard (1964)

Painlevé transcendents and their appearance in physics

- **Impenetrable Bose gas** $g \to \infty$

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + g \sum_{i<j} \delta(z_i - z_j)$$

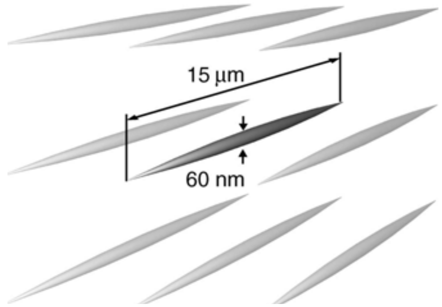


FIG. 1. The geometry and size of trapped 1D gases in a two-dimensional optical lattice. The spacing between the 1D tubes in the horizontal and vertical direction is 413 nm.

© T. Esslinger ETH Zürich

M. Jimbo, T. Miwa, Y. Môri, and M. Sato (1980)

$$\varrho_N(x) = N \int_0^L dz_2 \cdots dz_N \, \Psi^*(x, z_2, \cdots, z_N)\, \Psi(0, z_2, \cdots, z_N)$$

$$\varrho_\infty(x) = \lim_{N\to\infty} \varrho_N(x)\Big|_{L=N} = \exp\left(\int_0^{\pi x} \frac{dt}{t}\, \sigma_V(t)\right)$$

⬅ σP_V

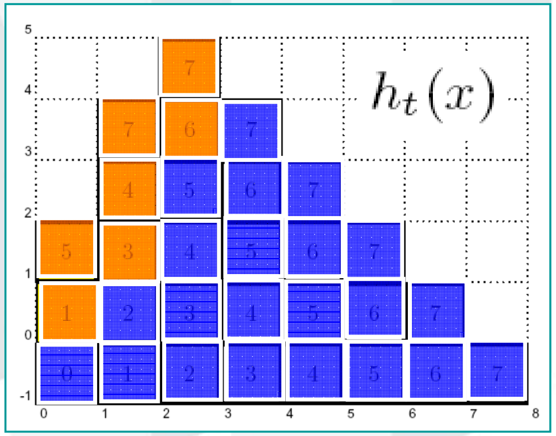
- **Painlevé transcendents and their appearance in physics**
  - **Growth models in (1+1)D / oriented digital boiling**

$$h_{t+1}(x) = \max\{h_t(x-1), h_t(x) + \varepsilon_{x,t}\}$$

$$h_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\varepsilon_{x,t} \sim \mathrm{Ber}(p)$$

J. Gravner, C. Tracy, and H. Widom (2001)



- **Painlevé transcendents and their appearance in physics**

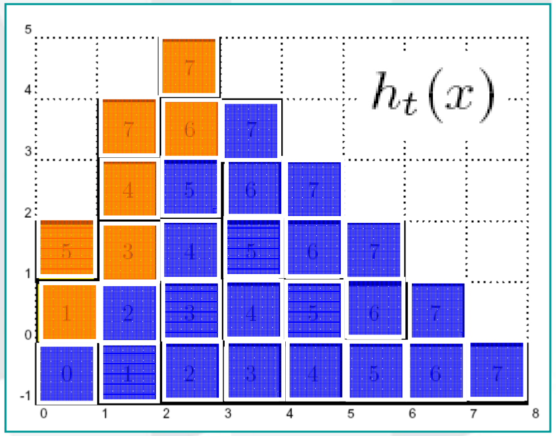
  - **Growth models in (1+1)D / oriented digital boiling**

$$h_{t+1}(x) = \max\left\{h_t(x-1), h_t(x) + \varepsilon_{x,t}\right\}$$

$$h_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\varepsilon_{x,t} \sim \mathrm{Ber}(p)$$

J. Gravner, C. Tracy, and H. Widom (2001)



Universal regime of shape fluctuations

$$x \to \infty,\ t \to \infty,\ p < p_c = 1 - x/t < 1$$

$$\mathrm{Prob}\left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} < s\right) = \boldsymbol{F}_2\left([\sigma_{\mathrm{II}}];\ s\right)$$

σP_{II}

▸ **Painlevé transcendents and their appearance in physics**

- **Growth models in (1+1)D / oriented digital boiling**

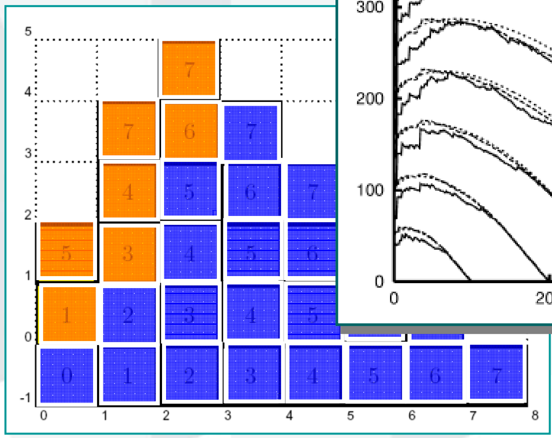
$h_{t+1}(x) = \mathrm{ma}$ …

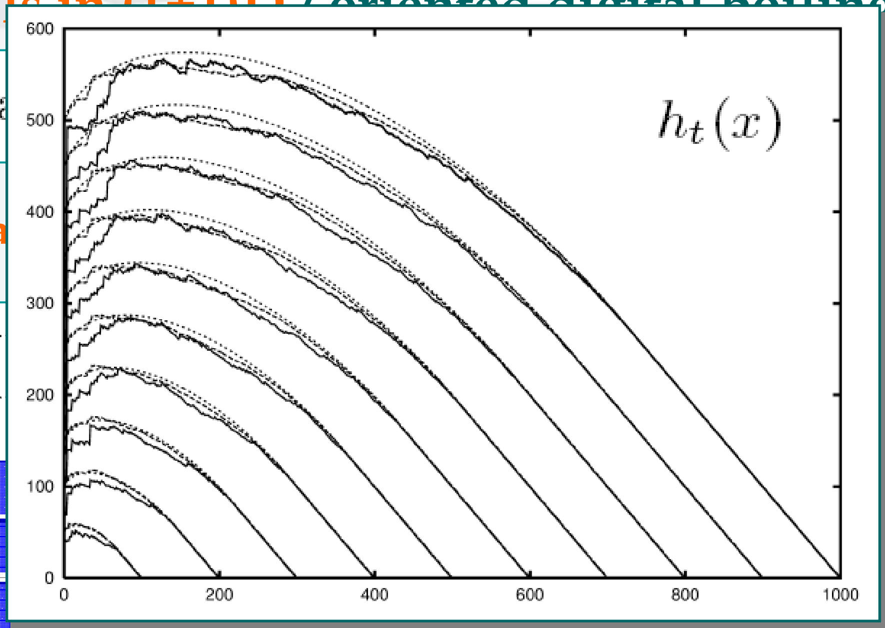
… 0 if $x = 0$

… $-\infty$ otherwise

… $\sim \mathrm{Ber}(p)$

J. Gravner, C. Tra…





…fluctuations

… $x/t < 1$

$\left(\dfrac{\cdots}{c_2 t^{1/3}} \right) = F_2\left([\sigma_{\mathrm{II}}]\,;\ s\right)$

σP_{II}

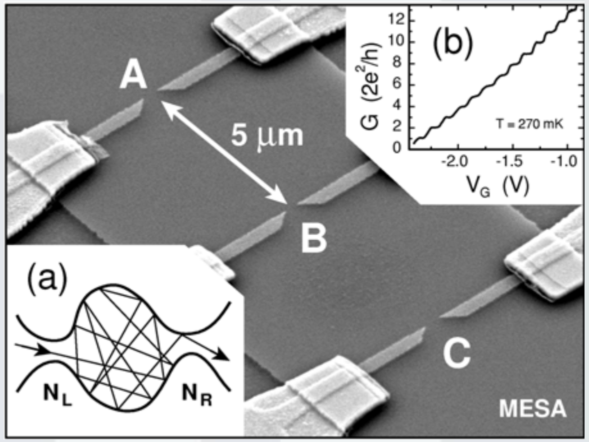
- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model ‣ 1D impenetrable Bose gas ‣ Growth models

- New! Painlevé transcendents in quantum transport problems: Announcement of the main result

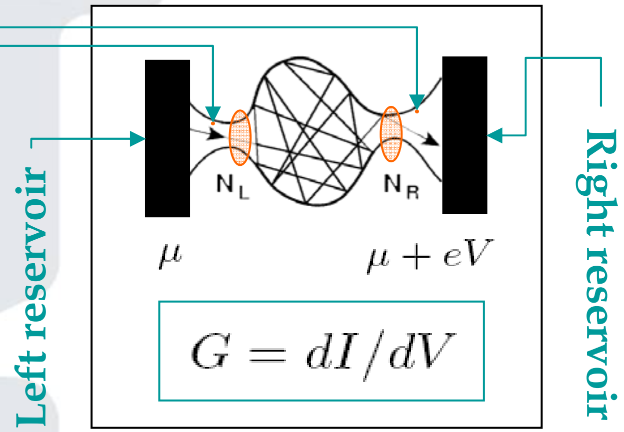
Painlevé transcendents in quantum transport problems

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell} \psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M} \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha} \chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k \right)$$

cavity — leads — coupling



© S. Oberholzer, Universität Bazel



Landauer conductance

- Painlevé transcendents in quantum transport problems

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

cavity — leads — coupling

Bohigas Conjecture

$\{\mathcal{H}_{k\ell}\} \in {\rm GUE}_{M\times M}$

+

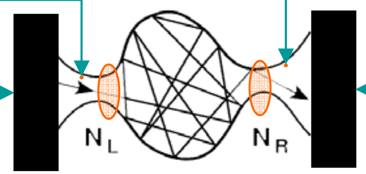
Quantum regime

$\tau_D \gg \tau_E$

Ehrenfest time

Electron dwell time

Left reservoir



Right reservoir

μ $\qquad$ $\mu + eV$

$G = dI/dV$

Landauer conductance

Painlevé Transcendents & Quantum Transport

- New: Painlevé in quantum transport !!

Painlevé transcendents in quantum transport problems

$$(t\sigma_V'')^2 - \left[\sigma_V - t\sigma_V' + 2(\sigma_V')^2 + (N_L + N_R)\sigma_V'\right]^2 + 4(\sigma_V')^2(\sigma_V' + N_L)(\sigma_V' + N_R) = 0$$

Main Result

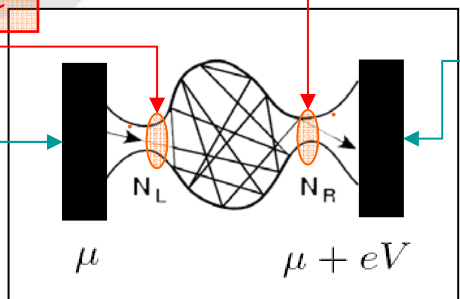
$$\sigma_V(z) = N_L N_R + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

σP_V

$$\kappa_\ell(g) = \langle\langle (G/G_0)^\ell \rangle\rangle$$

BPC

Left reservoir



$$G = dI/dV$$

Right reservoir

Landauer conductance

- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model
  - 1D impenetrable Bose gas
  - Growth models

- New! Painlevé transcendents in quantum transport problems: Announcement of the main result

- **Landauer conductance and its cumulants: Known results**

▸ Landauer conductance and its cumulants: Known results

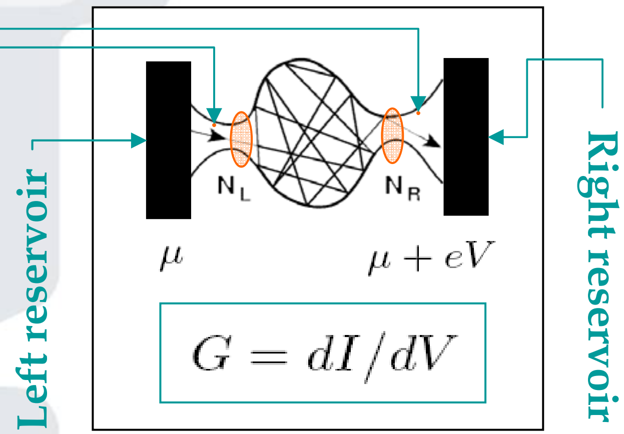
$$H_{\text{tot}} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\text{L}}+N_{\text{R}}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\text{L}}+N_{\text{R}}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

cavity — leads — coupling

Scattering matrix approach

$$G/G_0 = \text{tr}\left(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger\right)$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_{\text{L}}} & \\ & 0_{N_{\text{R}}} \end{pmatrix} \qquad \mathcal{C}_2 = \begin{pmatrix} 0_{N_{\text{L}}} & \\ & \mathbb{1}_{N_{\text{R}}} \end{pmatrix}$$

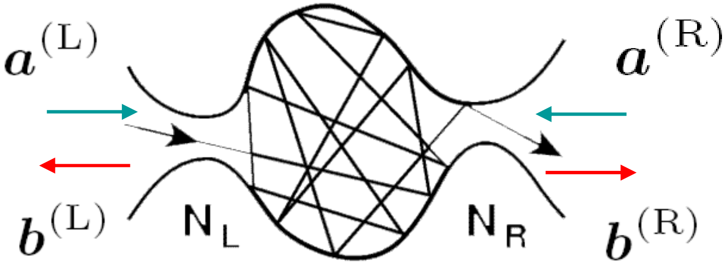


Landauer conductance

Landauer conductance and its cumulants: Known results

$$\mathcal{S}\mathcal{S}^\dagger = \mathbb{1}$$

$$\boldsymbol{a}^{(\mathrm{L})} = \begin{pmatrix} a_1^{(\mathrm{L})} \\ \vdots \\ a_{N_\mathrm{L}}^{(\mathrm{L})} \end{pmatrix} \qquad \boldsymbol{a}^{(\mathrm{R})} = \begin{pmatrix} a_1^{(\mathrm{R})} \\ \vdots \\ a_{N_\mathrm{R}}^{(\mathrm{R})} \end{pmatrix}$$



$$\boldsymbol{b}^{(\mathrm{L})} = \begin{pmatrix} b_1^{(\mathrm{L})} \\ \vdots \\ b_{N_\mathrm{L}}^{(\mathrm{L})} \end{pmatrix} \qquad \boldsymbol{b}^{(\mathrm{R})} = \begin{pmatrix} b_1^{(\mathrm{R})} \\ \vdots \\ b_{N_\mathrm{R}}^{(\mathrm{R})} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{b}^{(\mathrm{L})} \\ \boldsymbol{b}^{(\mathrm{R})} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \boldsymbol{a}^{(\mathrm{L})} \\ \boldsymbol{a}^{(\mathrm{R})} \end{pmatrix}$$

U_{N_R} $\mathbb{1}_{N_\mathrm{R}}$

Landauer conductance

Painlevé Transcendents & Quantum Transport

Landauer Conductance …

$$\mathcal{S}(\varepsilon_F) = \mathbb{1} - 2i\pi \mathcal{W}^\dagger (\varepsilon_F - \mathcal{H} + i\pi \mathcal{W}\mathcal{W}^\dagger)^{-1} \mathcal{W}$$

$$\mathcal{H} \in \mathrm{GUE}_{M\times M}$$

P. Brouwer (1995)

$$P(\mathcal{S}) \propto \left[\det\left(\mathbb{1} - \bar{\mathcal{S}}\mathcal{S}^\dagger\right) \det\left(\mathbb{1} - \mathcal{S}\bar{\mathcal{S}}^\dagger\right)\right]^{-(N_\mathrm{L}+N_\mathrm{R})}$$

Landau… …lts

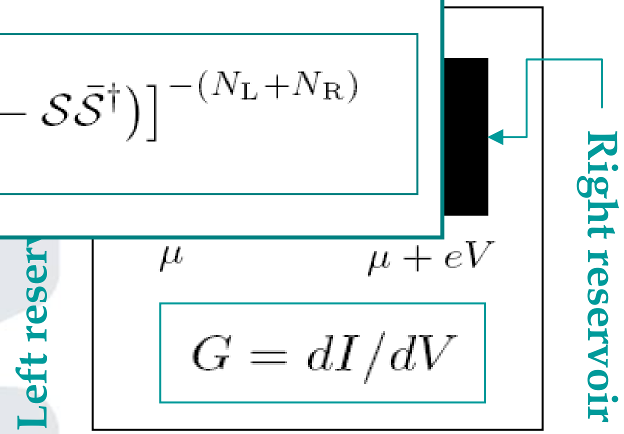
$$H_{\mathrm{tot}} = \sum_{k} \ldots \psi^\dagger_{k\alpha} \mathcal{W}^*_{k\alpha} \psi_k \Big)$$

…ng

Scatteri…

$$G/G_0 = \mathrm{tr}\left(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger\right)$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_\mathrm{L}} & \\ & 0_{N_\mathrm{R}} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_\mathrm{L}} & \\ & \mathbb{1}_{N_\mathrm{R}} \end{pmatrix}$$



Landauer conductance

Landauer conductance and its cumulants: Known results

$$H_{\text{tot}} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_L+N_R} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_L+N_R}\left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

$$\mathcal{S} \in \text{CUE}(N_L + N_R)$$

Scattering matrix approach

$$G/G_0 = \text{tr}\left(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger\right)$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_L} & \\ & 0_{N_R} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_L} & \\ & \mathbb{1}_{N_R} \end{pmatrix}$$

Semiclassical arguments:
Blümel & Smilansky (1990)

Microscopic justification:
Brouwer (1995)

Early (exact) calculation of moments/cumulants:
Baranger & Mello (1994)
1st & 2nd cumulants

Landauer conductance and its cumulants: Known results

$\sim \exp\left(\pi\sqrt{2n/3}\right)$

exponential growth !!

$$\langle g^n \rangle = n\,! \sum_{\lambda \vdash n} \frac{[N_1]_\lambda [N_2]_\lambda}{[N]_\lambda H_\lambda^2}$$

$$\mathcal{S} \in \mathrm{CUE}(N_{\mathrm{L}} + N_{\mathrm{R}})$$

All moments
Novaes (2008)

Symmetric functions

3^{rd} & 4^{th} cumulants
Savin, Sommers & Wieczorek (2007)

Selberg integral

Semiclassical arguments:
Blümel & Smilansky (1990)

Microscopic justification:
Brouwer (1995)

Early (exact) calculation of moments/cumulants:
Baranger & Mello (1994)
1^{st} & 2^{nd} cumulants

Outline

- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model
  - 1D impenetrable Bose gas
  - Growth models
- New! Painlevé transcendents in quantum transport problems: Announcement of the main result
- Landauer conductance and its cumulants: Known results
- **Integrable theory of quantum transport in chaotic cavities**
  - **Landauer conductance**
  - Noise power (thermal-to-shot-noise crossover)

‣ Integrable theory of quantum transport (Landauer conductance)

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

$$\mathcal{S} \in {\rm CUE}(N_{\rm L}+N_{\rm R})$$

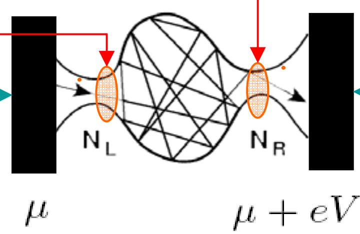
BPC

Scattering matrix approach

$$G/G_0 = {\rm tr}\left(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger\right)$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_{\rm L}} & \\ & 0_{N_{\rm R}} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_{\rm L}} & \\ & \mathbb{1}_{N_{\rm R}} \end{pmatrix}$$

Left reservoir



Right reservoir

$$G = dI/dV$$

Landauer conductance

‣ Integrable theory of quantum transport (Landauer conductance)

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in \mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

Cumulant generating function

$n = \min(N_\mathrm{L}, N_\mathrm{R})$

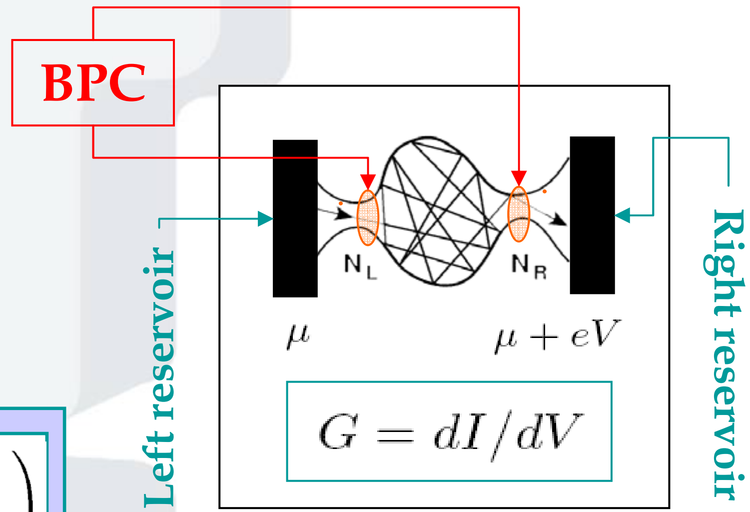
Itzykson-Zuber formula, but: high degeneracy of C-matrices

$$\mathcal{S} = \begin{pmatrix} r_{N_\mathrm{L}\times N_\mathrm{L}} & t'_{N_\mathrm{L}\times N_\mathrm{R}} \\ t_{N_\mathrm{R}\times N_\mathrm{L}} & r'_{N_\mathrm{R}\times N_\mathrm{R}} \end{pmatrix}$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_\mathrm{L}} & \\ & 0_{N_\mathrm{R}} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_\mathrm{L}} & \\ & \mathbb{1}_{N_\mathrm{R}} \end{pmatrix}$$

Truncate!

(Zyczkowski & Sommers, 2000)



Landauer conductance

▸ Integrable theory of quantum transport (Landauer conductance)

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

Cumulant generating function

$$n = \min(N_\mathrm{L}, N_\mathrm{R})$$

$$\nu = |N_\mathrm{L} - N_\mathrm{R}|$$

Itzykson-Zuber formula, but: high degeneracy of C-matrices

$$\mathcal{S} = \begin{pmatrix} r_{N_\mathrm{L}\times N_\mathrm{L}} & t'_{N_\mathrm{L}\times N_\mathrm{R}} \\ t_{N_\mathrm{R}\times N_\mathrm{L}} & r'_{N_\mathrm{R}\times N_\mathrm{R}} \end{pmatrix}$$

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\boldsymbol{t}\boldsymbol{t}^\dagger)} \right\rangle_{\boldsymbol{t}_{N_\mathrm{R}\times N_\mathrm{L}}}$$

Truncate!

(Zyczkowski & Sommers, 2000)

$$\mathcal{F}_n(z) \propto \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^\nu \exp(-zT_j) \cdot \Delta_n^2(\boldsymbol{T})$$

$$\mathcal{S} \in \mathrm{CUE}(N_\mathrm{L} + N_\mathrm{R})$$

Integrable theory of quantum transport (Landauer conductance)

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

Cumulant generating function

$$n = \min(N_\mathrm{L}, N_\mathrm{R})$$

$$\nu = |N_\mathrm{L} - N_\mathrm{R}|$$

$$\mathcal{F}_n(z) = \exp\left(\int_0^z dt \frac{\sigma_\mathrm{V}(t) - N_\mathrm{L}N_\mathrm{R}}{t}\right)$$

$$\mathcal{F}_n(z) \propto z^{-n(n+\nu)} \int_{(0,z)^n} \prod_{j=1}^n d\lambda_j\, \lambda_j^\nu\, e^{-\lambda_j} \cdot \Delta_n^2(\boldsymbol{\lambda})$$

Gap formation probability (LUE)
Tracy & Widom (1994)

$$(t\sigma_\mathrm{V}'')^2 - \left[\sigma_\mathrm{V} - t\sigma_\mathrm{V}' + 2(\sigma_\mathrm{V}')^2 + (N_\mathrm{L}+N_\mathrm{R})\sigma_\mathrm{V}'\right]^2 + 4(\sigma_\mathrm{V}')^2(\sigma_\mathrm{V}' + N_\mathrm{L})(\sigma_\mathrm{V}' + N_\mathrm{R}) = 0$$

▸ **Integrable theory of quantum transport (Landauer conductance)**

Cumulant generating function

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

$$\mathcal{F}_n(z) = \exp\left(\int_0^z dt\, \frac{\sigma_\mathrm{V}(t) - N_\mathrm{L}N_\mathrm{R}}{t}\right)$$

$$\sigma_\mathrm{V}(z) = N_\mathrm{L}N_\mathrm{R} + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\!\langle (G/G_0)^\ell \rangle\!\rangle$$

!

$$\log \mathcal{F}_n(z) = \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell+1)} \langle\!\langle (G/G_0)^\ell \rangle\!\rangle$$

Main Result

$$(t\sigma_\mathrm{V}'')^2 - \left[\sigma_\mathrm{V} - t\sigma_\mathrm{V}' + 2(\sigma_\mathrm{V}')^2 + (N_\mathrm{L}+N_\mathrm{R})\sigma_\mathrm{V}'\right]^2 + 4(\sigma_\mathrm{V}')^2(\sigma_\mathrm{V}'+N_\mathrm{L})(\sigma_\mathrm{V}'+N_\mathrm{R}) = 0$$

Painlevé Transcendents & Quantum Transport

- New: Painlevé in quantum transport !!

- Painlevé transcendents in quantum transport problems

$$H_{\text{tot}} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell} \psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M} \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha} \chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k \right)$$

cavity — leads — coupling

Bohigas Conjecture

$\{\mathcal{H}_{k\ell}\} \in \text{GUE}_{M\times M}$

+

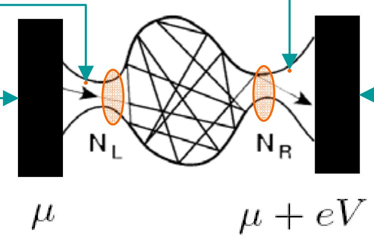
Quantum regime

$\tau_D \gg \tau_E$

Ehrenfest time

Electron dwell time

Left reservoir



$G = dI/dV$

Right reservoir

Landauer conductance

Painlevé transcendents in quantum transport problems

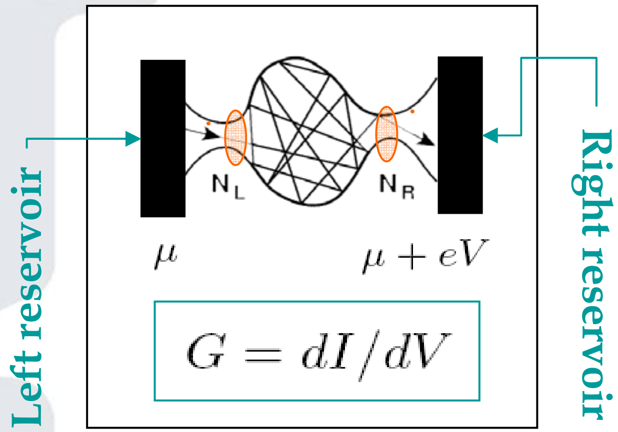
$$(t\sigma_{\rm V}'')^2 - \left[\sigma_{\rm V} - t\sigma_{\rm V}' + 2(\sigma_{\rm V}')^2 + (N_{\rm L}+N_{\rm R})\sigma_{\rm V}'\right]^2 + 4(\sigma_{\rm V}')^2(\sigma_{\rm V}'+N_{\rm L})(\sigma_{\rm V}'+N_{\rm R}) = 0$$

Main Result

$$\sigma_{\rm V}(z) = N_{\rm L}N_{\rm R} + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

σP$_{\rm V}$

$$\kappa_\ell(g) = \langle\langle (G/G_0)^\ell \rangle\rangle$$



Landauer conductance

Consequences / further results

- Conductance cumulants obey a nonlinear recurrence equation

$$[(N_{\rm L}+N_{\rm R})^2-j^2](j+1)\kappa_{j+1}(g) = 2\sum_{\ell=0}^{j-1}(3\ell+1)(j-\ell)^2\binom{j}{\ell}\kappa_{\ell+1}(g)\kappa_{j-\ell}(g) - (N_{\rm L}+N_{\rm R})(2j-1)\,j\kappa_j(g)-j(j-1)(j-2)\,\kappa_{j-1}(g)$$

$$\kappa_1(g)=\frac{N_{\rm L}N_{\rm R}}{N_{\rm L}+N_{\rm R}}$$

$$\kappa_2(g)=\frac{\kappa_1^2(g)}{(N_{\rm L}+N_{\rm R})^2-1}$$

$\Rightarrow$

$$\kappa_\ell(g)=P_\ell\left(\kappa_1(g);N_{\rm L}+N_{\rm R}\right)$$

$$\langle g^n\rangle = n\,!\sum_{\lambda\vdash n}\frac{[N_1]_\lambda[N_2]_\lambda}{[N]_\lambda H_\lambda^2}$$

$$\sim \exp\left(\pi\sqrt{2n/3}\right)$$

Novaes (2008)

Consequences / further results

- Entire conductance distribution function follows from the Toda Lattice

$$\mathcal{F}_n(z)\,\mathcal{F}_n''(z) - \left(\mathcal{F}_n'(z)\right)^2 = \mathrm{var}_n(g)\,\mathcal{F}_{n-1}(z)\,\mathcal{F}_{n+1}(z)$$

$$\mathcal{F}_0(z) = 0$$

$$\mathcal{F}_1(z) = \frac{(\nu+1)!}{z^{\nu+1}}\left(1 - e^{-z}\sum_{\ell=0}^{\nu}\frac{z^\ell}{\ell!}\right)$$

- Asymptotic analysis of conductance cumulants
- Asymptotic analysis of conductance distribution (deviations from the Gaussian law)

Consequences / further results

- Entire conductance distribution function follows from the Toda Lattice

$$\mathcal{F}_n(z)\,\mathcal{F}_n''(z) - \left(\mathcal{F}_n'(z)\right)^2 = \mathrm{var}_n(g)\,\mathcal{F}_{n-1}(z)\,\mathcal{F}_{n+1}(z)$$

$$\mathcal{F}_0(z) = 0$$

$$\mathcal{F}_1(z) = \frac{(\nu+1)!}{z^{\nu+1}}\left(1 - e^{-z}\sum_{\ell=0}^{\nu}\frac{z^\ell}{\ell!}\right)$$

$$f_n(g) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\,\mathcal{F}_n(z)\,e^{g\,z}$$

Conductance probability density function

- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model
  - 1D impenetrable Bose gas
  - Growth models

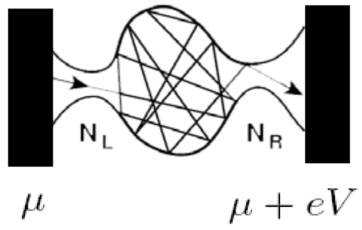
Statistics of thermal to shot noise crossover in chaotic cavities

Eugene Kanzieper[1] and Vladimir Al. Osipov[2]
[1] *Department of Applied Mathematics, H.I.T.—Holon Institute of Technology, Holon 58102, Israel*
[2] *Fachbereich Physik, Universität Duisburg-Essen, D-47057 Duisburg, Germany*
(Dated: December 20, 2008)

- **Integrable theory** of quantum transport in chaotic cavities
  - Landauer conductance
  - Noise power (thermal-to-shot-noise crossover)

Integrable theory of quantum transport (Noise power)

- Charge transfer through a quantum chaotic cavity
  - discreteness of electron charge
  - quantum nature of electrons



$$G_0 = e^2/h$$

- Fluctuating current quantified by noise power

$$\mathcal{P} = 2\int_{-\infty}^{+\infty} dt\, \langle \delta I(t+t_0)\delta I(t_0)\rangle_{t_0}$$

- At both finite temperature and bias voltage

$$\mathcal{P}(\theta, v) = 4\theta\, G_0 \left(\mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2 + \frac{v}{2\theta} \coth\left(\frac{v}{2\theta}\right) \left[\mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger) - \mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2 \right] \right)$$

Integrable theory of quantum transport (Noise power)

- At temperatures larger than bias voltage $(\theta \gg v)$ equilibrium thermal noise dominates:

$$\mathcal{P}_{\mathrm{th}}(\theta) = 4\theta\, G_0\, \mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger).$$

- At low temperatures $(\theta \ll v)$ nonequilibrium current fluctuations – shot noise – dominates:

$$\mathcal{P}_{\mathrm{shot}}(v) = 4v\, G_0 \left[\mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger) - \mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2\right].$$

- At both finite temperature and bias voltage

$$\mathcal{P}(\theta, v) = 4\theta\, G_0 \left(\mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2 + \frac{v}{2\theta} \coth\left(\frac{v}{2\theta}\right) \left[\mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger) - \mathrm{tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2\right]\right)$$

Thermal to shot noise crossover beyond the average noise power

$$\langle \mathcal{P}(\theta, \upsilon) \rangle_{\mathcal{S}} = \langle \mathcal{P}_{\rm th} \rangle_{\mathcal{S}} \left[1 + \frac{N_{\rm L} N_{\rm R}}{(N_{\rm L}+N_{\rm R})^2 - 1} f_\beta \right],$$

where

$$\langle \mathcal{P}_{\rm th} \rangle_{\mathcal{S}} = 4\theta\, G_0 \frac{N_{\rm L} N_{\rm R}}{N_{\rm L}+N_{\rm R}}$$

is the average equilibrium thermal noise power, and the thermodynamic function

$$f_\beta = \beta \coth \beta - 1$$

is taken at $\beta = \upsilon/2\theta$.

Theory:
Blanter & Sukhorukov (2000)

Experiment:
Oberholzer *et al* (2001)

- **At both finite temperature and bias voltage**

$$\mathcal{P}(\theta, \upsilon) = 4\theta\, G_0 \left({\rm tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2 + \frac{\upsilon}{2\theta} \coth\left(\frac{\upsilon}{2\theta}\right) \left[{\rm tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger) - {\rm tr}(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger)^2 \right] \right)$$

Thermal to shot noise crossover beyond the average noise power

- **Joint cumulant generating function**

$$\mathcal{F}_n(z,w) = \langle \exp(-z\, G/G_0)\, \exp(-w\, \mathcal{P}/\mathcal{P}_0)\rangle_{\mathcal{S}\in \mathrm{CUE}(N)}$$

- **Truncated scattering matrix:**

$$\mathcal{S} = \begin{pmatrix} r_{N_\mathrm{L}\times N_\mathrm{L}} & t'_{N_\mathrm{L}\times N_\mathrm{R}} \\ t_{N_\mathrm{R}\times N_\mathrm{L}} & r'_{N_\mathrm{R}\times N_\mathrm{R}} \end{pmatrix}$$

No longer a gap formation probability !!

$$\mathcal{F}_n(z,w) = c_n^{-1} \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^{\nu}\, \Gamma_{z,w}(T_j)\, \Delta_n^2(\boldsymbol{T}),$$

where

$$\Gamma_{z,w}(T) = \exp\left[-(z+w)T - w\, f_\beta\, T(1-T)\right].$$

$$n = \min(N_\mathrm{L}, N_\mathrm{R})$$

$$\nu = |N_\mathrm{L} - N_\mathrm{R}|$$

Thermal to shot noise crossover beyond the average noise power

- Joint cumulant generating function

$$\log \mathcal{F}_n(z,w) = \sum_{\ell,m=0}^{\infty} (-1)^{\ell+m} \frac{z^\ell w^m}{\ell!\, m!} \kappa_{\ell,m}, \qquad \kappa_{\ell,m} = \langle\!\langle (G/G_0)^\ell (\mathcal{P}/\mathcal{P}_0)^m \rangle\!\rangle,$$

- More sophisticated methods needed… (similar to maths of bosonic replicas)

$$w f_\beta^2 \frac{\partial^4}{\partial z^4} \log \mathcal{F}_n(z,w) + 6w\, f_\beta^2 \left(\frac{\partial^2}{\partial z^2} \log \mathcal{F}_n(z,w) \right)^2 + 2 \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z} \right) \log \mathcal{F}_n(z,w)$$
$$+ \left(\left[2(2n+\nu) f_\beta - 2z + w\,(1-f_\beta^2) \right] \frac{\partial^2}{\partial z^2} + 2(z-2w) \frac{\partial^2}{\partial z \partial w} + 3w \frac{\partial^2}{\partial w^2} \right) \log \mathcal{F}_n(z,w) = 0.$$

- Nonlinear recurrence equation for joint cumulants

Thermal to shot noise crossover beyond the average noise power

- Joint cumulant generating function

$$\log \mathcal{F}_n(z,w) = \sum_{\ell,m=0}^{\infty} (-1)^{\ell+m} \frac{z^\ell w^m}{\ell!\, m!} \kappa_{\ell,m}, \qquad \kappa_{\ell,m} = \langle\langle (G/G_0)^\ell (\mathcal{P}/\mathcal{P}_0)^m \rangle\rangle,$$

- More sophisticated methods needed… (similar to maths of bosonic replicas)

$$\mathrm{var}\left[\mathcal{P}(v,\theta)\right] = \kappa_{0,2} = \frac{\theta^2}{16} + \frac{1}{128}\left[v \coth\left(\frac{v}{\theta}\right) - \theta\right]^2.$$

Limit of large number of channels (symmetric leads)

- Nonlinear recurrence equation for joint cumulants

- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model
  - 1D impenetrable Bose gas
  - Growth models
- New! Painlevé transcendents in quantum transport problems: Announcement of the main result
- Landauer conductance and its cumulants: Known results
- Integrable theory of quantum transport in chaotic cavities
  - Landauer conductance
  - Noise power (thermal-to-shot-noise crossover)
- **Conclusions / Open problems**

Conclusions / Open problems

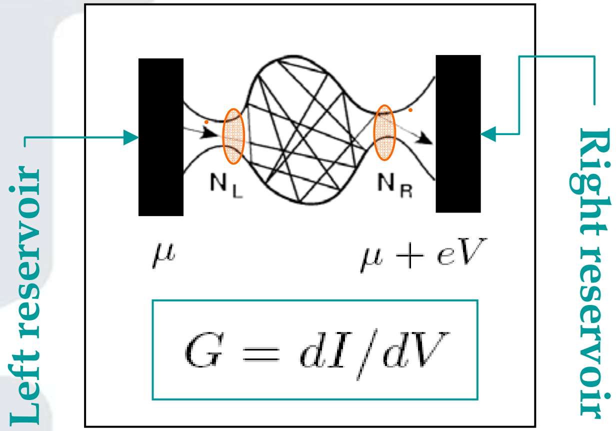
$$(t\sigma_V'')^2 - \left[\sigma_V - t\sigma_V' + 2(\sigma_V')^2 + (N_L + N_R)\sigma_V'\right]^2 + 4(\sigma_V')^2(\sigma_V' + N_L)(\sigma_V' + N_R) = 0$$

Main Result

$$\sigma_V(z) = N_L N_R + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

σP_V

$$\kappa_\ell(g) = \langle\langle (G/G_0)^\ell \rangle\rangle$$



Landauer conductance

Conclusions / Open problems

- Non-ideal contacts (Poisson kernel)

$$P(\mathcal{S}) \propto \left[\det\left(\mathbb{1} - \bar{\mathcal{S}}\mathcal{S}^\dagger\right)\det\left(\mathbb{1} - \mathcal{S}\bar{\mathcal{S}}^\dagger\right)\right]^{-(N_{\rm L}+N_{\rm R})}$$

- Lossy quantum transport (electrons escaping through the third lead)
- Full counting statistics
- Other symmetry classes (β=1 and β=4)

Painlevé Transcendents and Quantum Transport

Eugene Kanzieper[1] and **Vladimir Al. Osipov**[2]

[1] H.I.T. – Holon Institute of Technology, Israel

[2] Universität Duisburg-Essen, Germany

ISF Research Workshop on Random Matrices and Integrability, Yad Hashmona, March 26, 2009