Integrable random matrices

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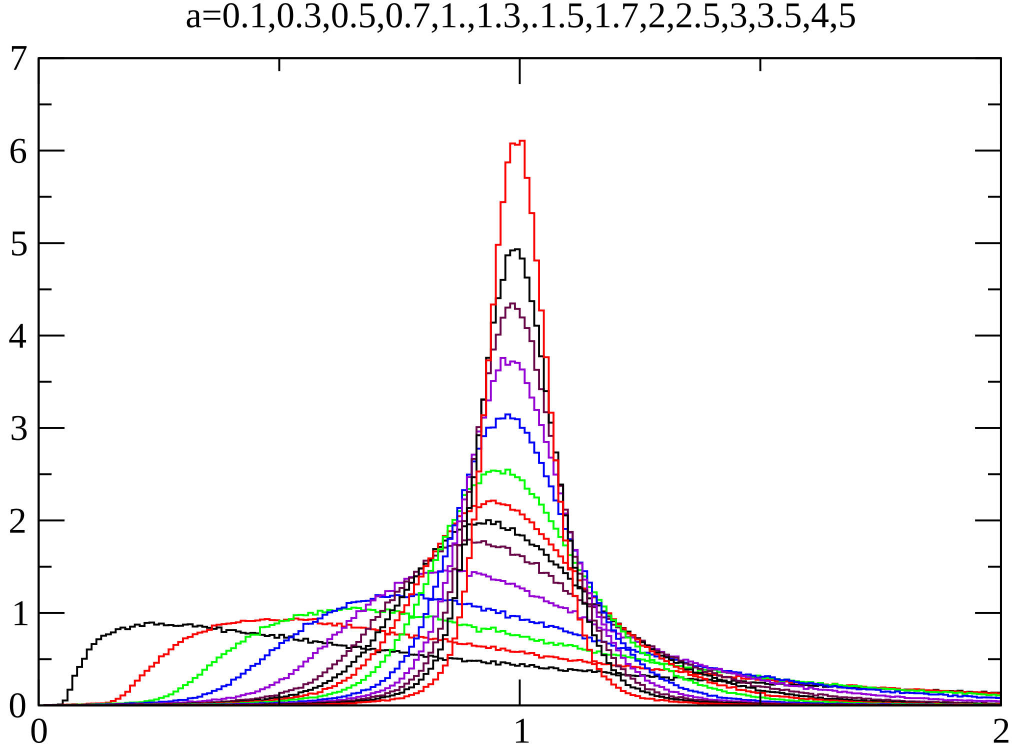
$$\mathbf{L_{kr} = p_r \delta_{kr} + ia \frac{1 - \delta_{kr}}{k - r}}$$

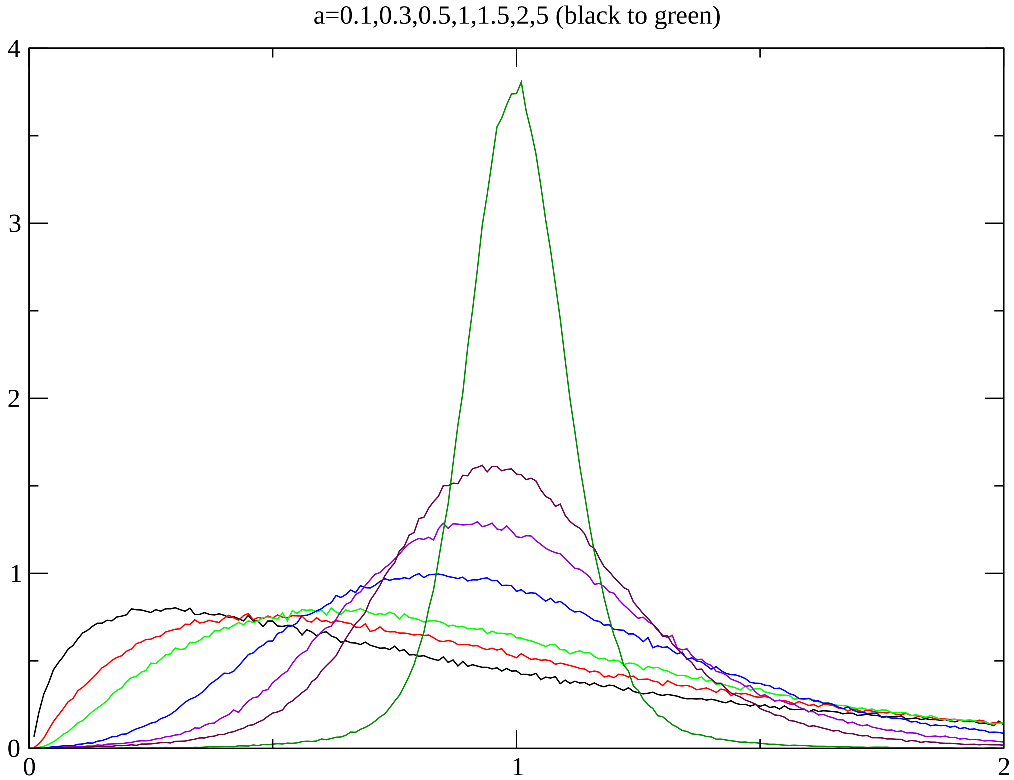
p_r are i.i.d. r. v. uniformly distributed between -1 and 1.

k, r are integers between $-N$ and N with step 2.

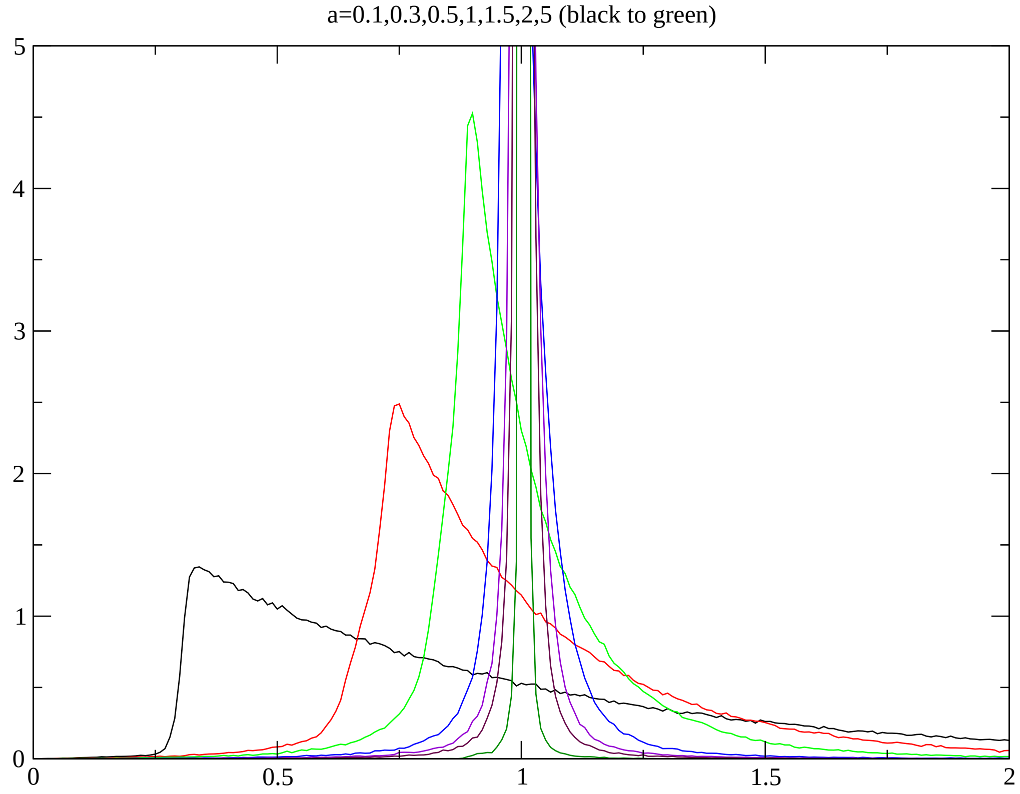


$$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1}-\delta_{\mathbf{kr}})}{\mathbf{2}\sinh(\mu(\mathbf{k}-\mathbf{r})/\mathbf{2})}$$

$$\mu = 2\pi/N$$



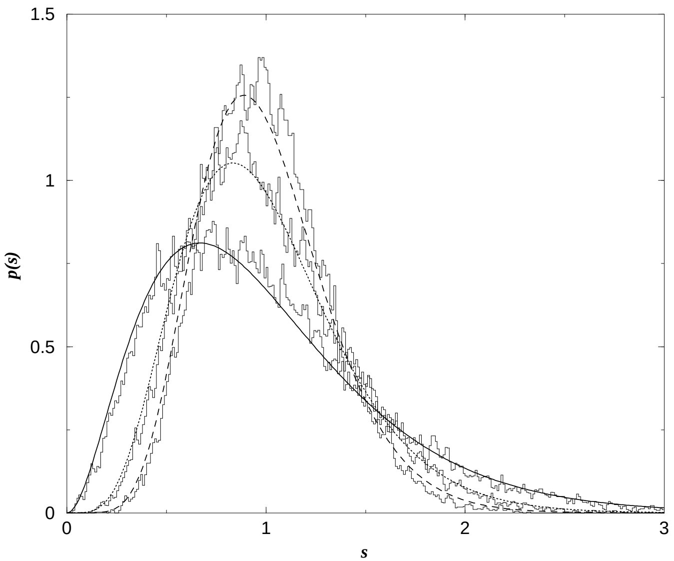
$$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1} - \delta_{\mathbf{kr}})}{\mathbf{2}\sin(\mu(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2})}$$



$$\mathbf{M_{kp}} = e^{i\mathbf{\Phi_k}} \frac{\mathbf{1} - e^{\mathbf{2}\pi i \alpha \mathbf{N}}}{\mathbf{N}[\mathbf{1} - e^{\mathbf{2}\pi i(\mathbf{k}-\mathbf{p}+\alpha\mathbf{N})/\mathbf{N}}]}$$

Φ_k are **i.i.d. r.v.** uniformly distributed between 0 and 2π.

$\alpha = m/q$ correspond to a quantization of an interval exchange map.



$\alpha = 1/3, 1/6, 1/9$.

$\mathbf{N} \equiv \pm\mathbf{1} \mod q$

Solid line: $\alpha = 1/3$

$p(s) \sim s^2\, e^{-3s}$

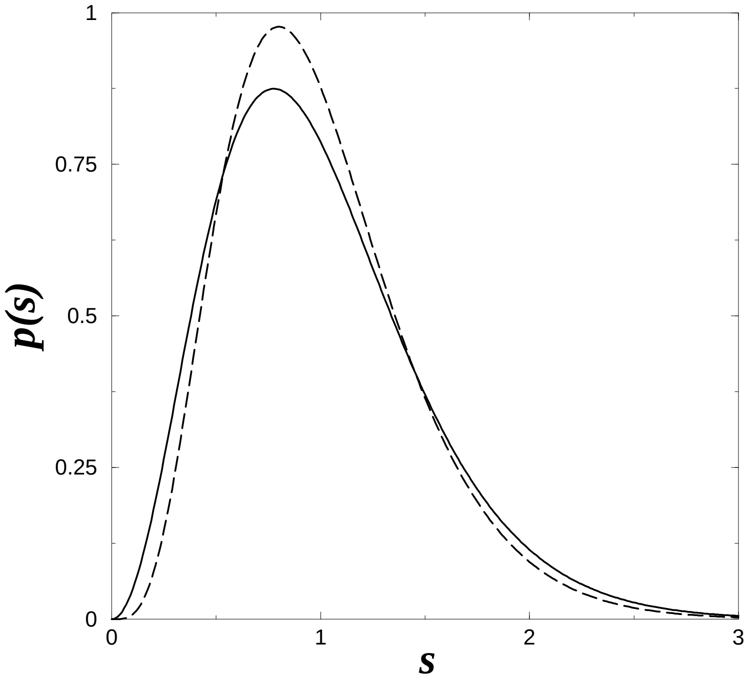
Dotted line: $\alpha = 1/6$

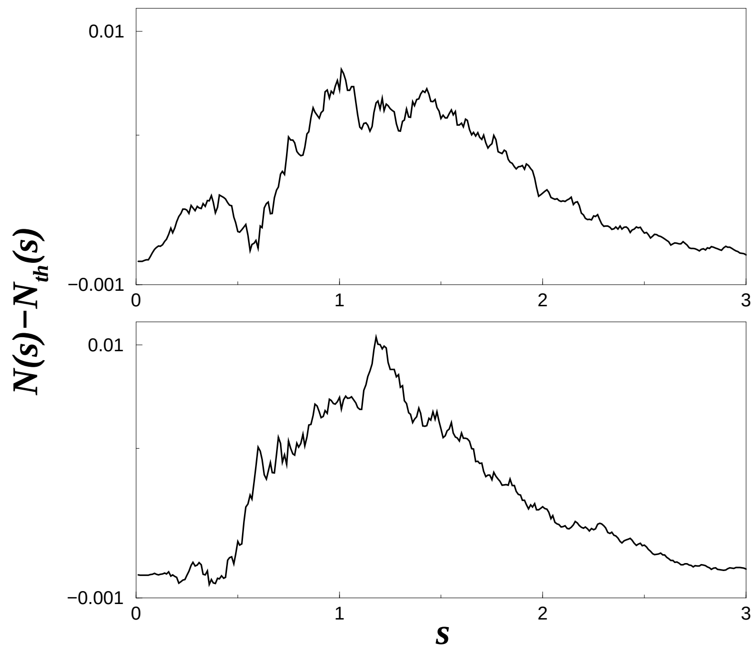
$p(s) \sim s^5\, e^{-6s}$

Dashed line: $\alpha = 1/9$

$p(s) \sim s^8\, e^{-9s}$

$\alpha = \mathbf{1/5}$





Left: Dashed line $\mathbf{N} \equiv \pm\mathbf{1}$ mod 5: $p(s) \sim s^4 \mathrm{e}^{-5s}$.

Solid line $\mathbf{N} \equiv \pm\mathbf{2}$ mod 5: $p(s)(a_2 s^2 + a_3 s^3 + a_4 s^4 + a_5 s^5 + a_6 s^6) e^{-5s}$

$a_2 \approx 5.041$, $a_3 \approx 25.203$, $a_4 \approx 45.724$, $a_5 \approx 32.451$, $a_6 \approx 8.357$.

Right: Differences $N(s) - N_{th}(s)$ for $N = 801$ and $N = 802$.

$$\mathbf{M_{kp}} = \mathrm{e}^{\mathrm{i}\boldsymbol{\Phi}_{\mathbf{k}}} \frac{\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi \mathrm{i}\alpha\mathbf{N}}}{\mathbf{N}[\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi\mathrm{i}(\mathbf{k}-\mathbf{p}+\alpha\mathbf{N})/\mathbf{N}}]}$$

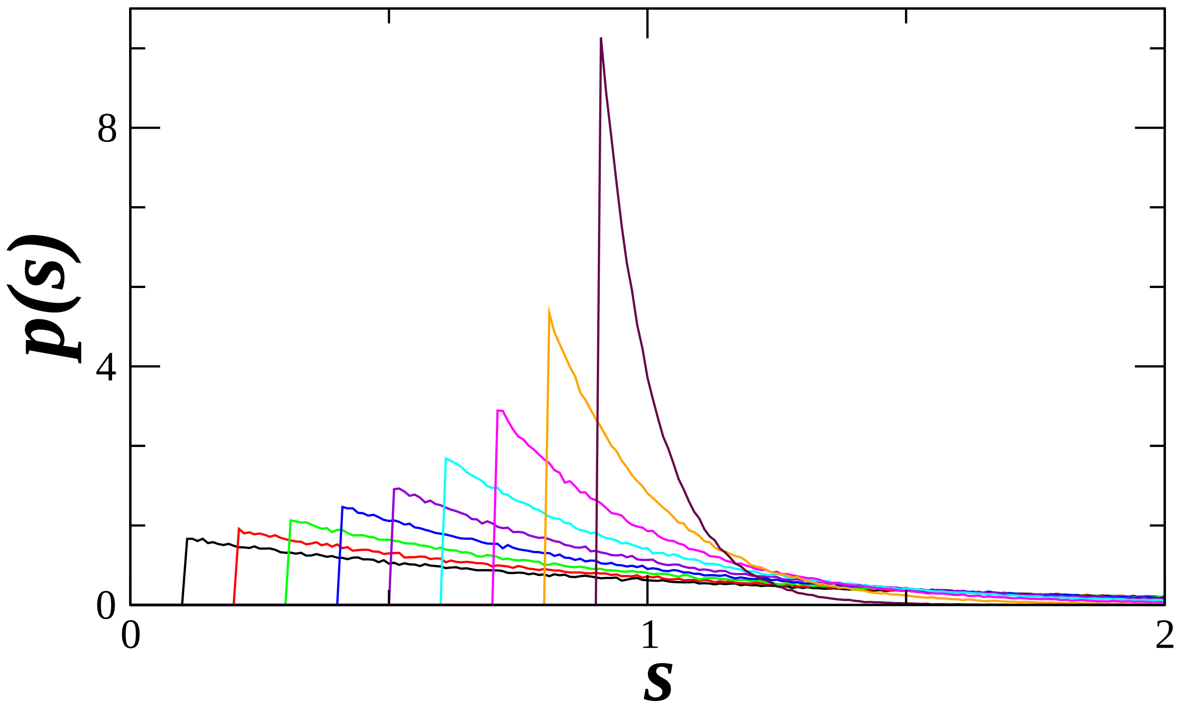
Substitution

$$\alpha = \frac{\mathbf{a}}{\mathbf{N}}$$

$$\mathbf{M_{kp}} = \mathrm{e}^{\mathrm{i}\boldsymbol{\Phi}_{\mathbf{k}}} \frac{\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi \mathrm{i}\mathbf{a}}}{\mathbf{N}[\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi\mathrm{i}(\mathbf{k}-\mathbf{p}+\mathbf{a})/\mathbf{N}}]}$$

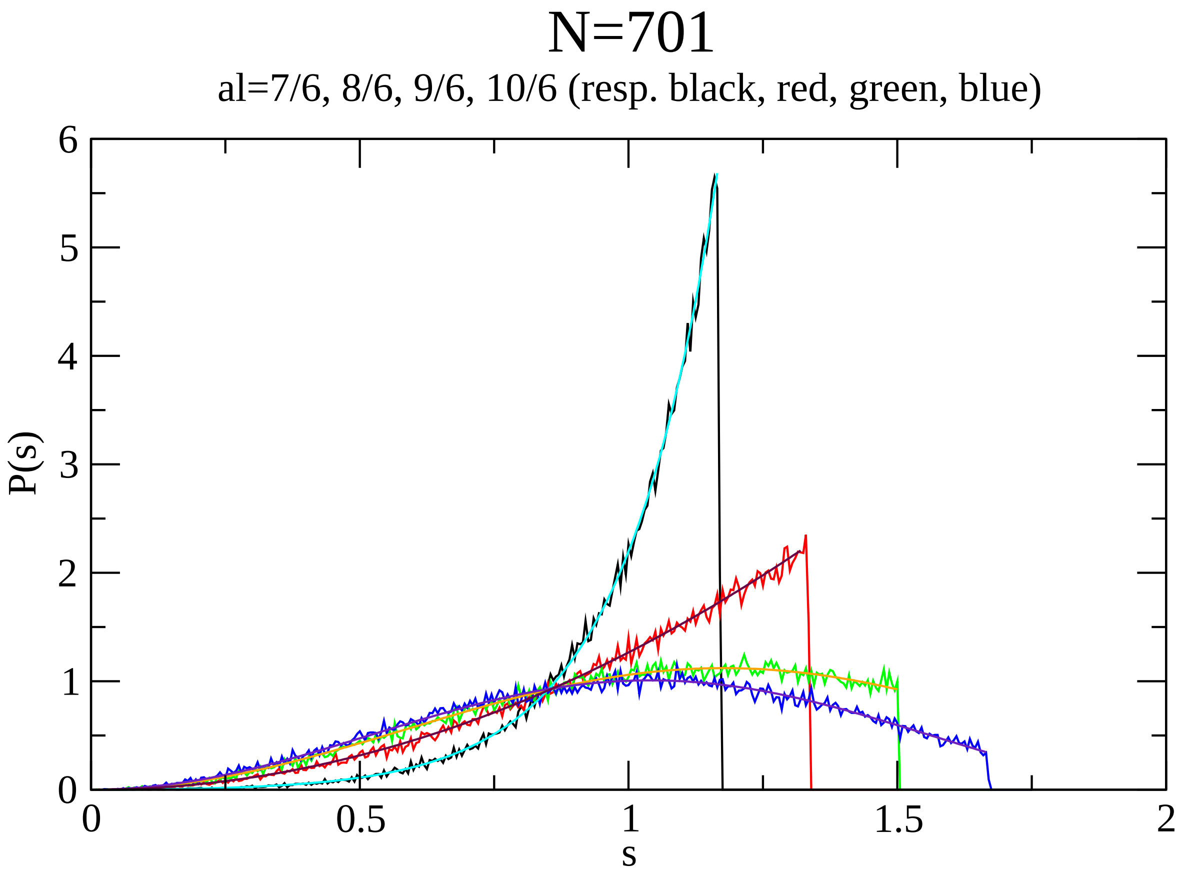
$\mathbf{0 < a < 1}$

P(s) for IIIb



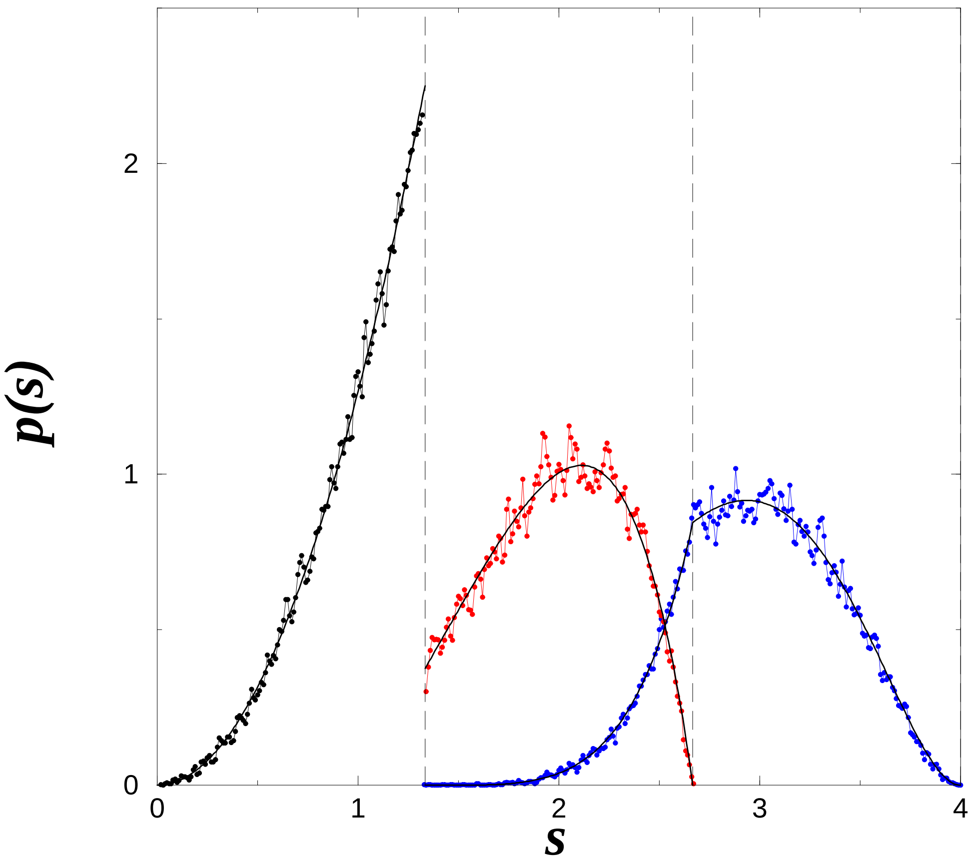
$N = 701$ and 1000 iterations. $\mathbf{a} = 0.1, 0.2, \ldots, 0.9$.

$\mathbf{1 < a < 2}$



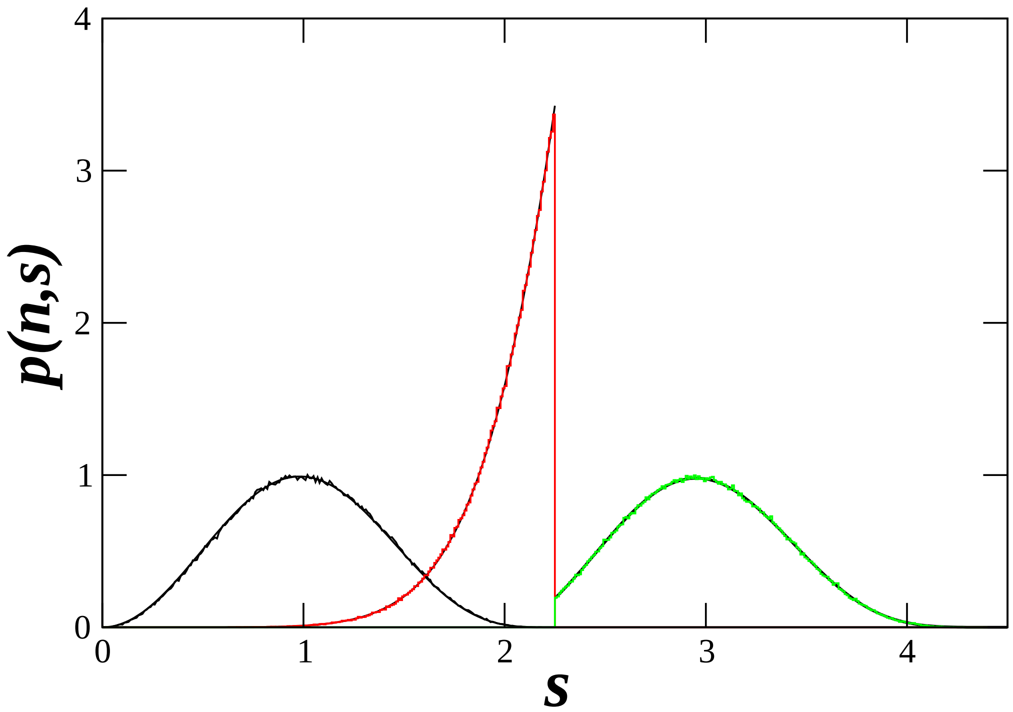
N=701
al=7/6, 8/6, 9/6, 10/6 (resp. black, red, green, blue)
P(s)
s
0
1
2
3
4
5
6
0
0.5
1
1.5
2

$\mathbf{a = 4/3}$



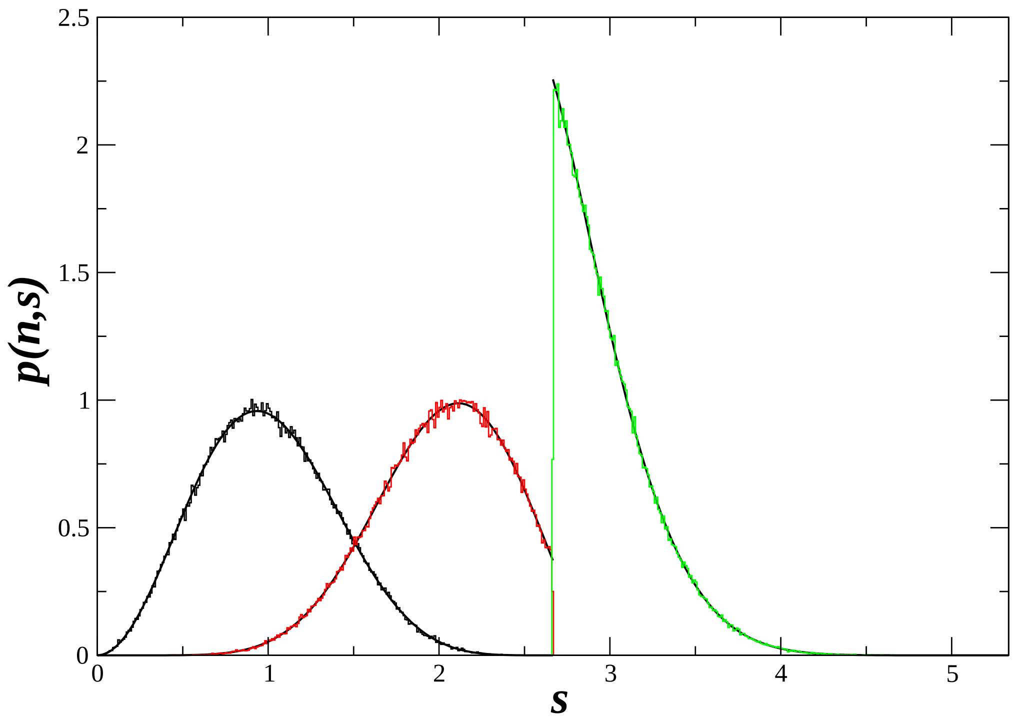
$N = 201$, 100 iterations

$$\mathbf{a = 9/4}$$



$N = 701$, 1000 iterations

$\mathbf{a = 16/3}$



$N = 701$, 1000 iterations

Generalizations

(Inr) $$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mathbf{1} - \delta_{\mathbf{kr}}}{\mathbf{k} - \mathbf{r}} \longrightarrow \mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mathbf{1} - \delta_{\mathbf{kr}}}{\mathbf{q_k} - \mathbf{q_r}}$$

(IInr) $$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1} - \delta_{\mathbf{kr}})}{\mathbf{2}\sinh(\mu(\mathbf{k} - \mathbf{r})/\mathbf{2})} \longrightarrow$$

$$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1} - \delta_{\mathbf{kr}})}{\mathbf{2}\sinh(\mu(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2})}$$

(IIInr) $$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1} - \delta_{\mathbf{kr}})}{\mathbf{2}\sin(\mu(\mathbf{k} - \mathbf{r})/\mathbf{2})} \longrightarrow$$

$$\mathbf{L_{kr}} = \mathbf{p_r}\delta_{\mathbf{kr}} + \mathrm{i}\mathbf{a}\frac{\mu(\mathbf{1} - \delta_{\mathbf{kr}})}{\mathbf{2}\sin(\mu(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2})}$$

$$\textbf{IIIb} \qquad \mathbf{M_{kp}} = \mathrm{e}^{\mathrm{i}\boldsymbol{\Phi}_\mathbf{k}} \frac{\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi\mathrm{i}\mathbf{a}}}{\mathbf{N}[\mathbf{1} - \mathrm{e}^{\mathbf{2}\pi\mathrm{i}(\mathbf{k}-\mathbf{p}+\mathbf{a})/\mathbf{N}}]} \longrightarrow$$

$$\mathbf{L_{kp}} = \mathrm{e}^{-\mathrm{i}\pi(\mathbf{N}-\mathbf{1})\mathbf{a}} \mathrm{e}^{\mathrm{i}\boldsymbol{\Phi}_\mathbf{k} + \mathrm{i}(\mathbf{q_k}-\mathbf{q_r})/\mathbf{2}} \mathbf{C_{kp}}$$

and C is an **orthogonal** matrix

$$C_{kp} = V_k^{1/2}(a;q) \frac{\sin \pi \mathbf{a}}{\sin\left(\frac{\mathbf{q_k}-\mathbf{q_p}}{2} + \pi\mathbf{a}\right)} V_p^{1/2}(-a;q)\ .$$

$$V_j(a,q) = \prod_{s\neq j} \frac{\sin\left(\frac{\mathbf{q_j}-\mathbf{q_s}}{2} + \pi\mathbf{a}\right)}{\sin\left(\frac{\mathbf{q_j}-\mathbf{q_s}}{2}\right)}\ ,$$

When $q_k = 2\pi k/N$, $L_{kp} = M_{kp}$

Classical integrable systems

Calogero models

Inr
$$H(p,q)=\sum_{j=1}^{N}\frac{1}{2}p_j^2+a^2\sum_{1\le j<k\le N}\frac{1}{(q_j-q_k)^2}$$

IInr
$$H(p,q)=\sum_{j=1}^{N}\frac{1}{2}p_j^2+\frac{1}{4}a^2\mu^2\sum_{1\le j<k\le N}\frac{1}{\sinh^2(\frac{\mu}{2}(q_j-q_k))}$$

IIInr
$$H(p,q)=\sum_{j=1}^{N}\frac{1}{2}p_j^2+\frac{1}{4}a^2\mu^2\sum_{1\le j<k\le N}\frac{1}{\sin^2(\frac{\mu}{2}(q_j-q_k)/2)}$$

Ruijsenaars model

IIIb
$$H(p,q)=\sum_{j=1}^{N}\cos(p_j)\prod_{k\neq j}\left(1-\frac{\sin^2\pi a}{\sin^2\frac{\mu}{2}(q_j-q_k)}\right)^{1/2}$$

Lax matrices

All $\mathbf{L_{kr}}$ are the Lax matrices of these integrable models

$$\dot{\mathbf{L}} = [\mathbf{L}\,\mathbf{M}]$$

For all integrable models there exist canonical variables **action-angle**

$$I_j = I_j(\vec{p},\vec{q}\,) \quad \phi_j = \phi_j(\vec{p},\vec{q}\,)$$

$$\mathrm{d}\vec{\mathbf{p}}\,\mathrm{d}\vec{\mathbf{q}} = \mathrm{d}\vec{\mathbf{I}}\,\mathrm{d}\vec{\phi}$$

Distribution of elements of random matrices $L_{kr}(\vec{p},\vec{q}\,)$

$$p(\vec{p},\vec{q}\,)\,\mathrm{d}\vec{\mathbf{p}}\,\mathrm{d}\vec{\mathbf{q}} = P(\vec{I},\vec{\phi}\,)\,\mathrm{d}\vec{\mathbf{I}}\,\mathrm{d}\vec{\phi}$$

Angle-action variables for the Calogero model

$$L_{kr} = p_r \delta_{kr} + \mathrm{i}g \frac{1-\delta_{kr}}{q_k - q_r} \ .$$

$$\sum_{r=1}^{N} L_{kr} u_r(n) = \lambda_n u_k(n), \ \ \sum_m u_k^*(m) u_r(m) = \delta_{kr}$$

One gets: $\quad L_{kr} q_r - q_k L_{kr} = -\mathrm{i}g(1-\delta_{kr}) \ .$

$$Q_{mn}(\lambda_m - \lambda_n) = -\mathrm{i}g(e_m^* e_n - \delta_{mn})$$

$$Q_{mn} = \sum_k u_k^*(m) q_k u_k(n), \ e_m = \sum_k u_k(m), \ \sum_n Q_{mn} u_k^*(n) = q_k u_k^*(m)$$

One can choose $e_m = 1$. Then

$$Q_{mn} = w_m \delta_{mn} - \mathrm{i}g \frac{1-\delta_{mn}}{\lambda_m - \lambda_n} \ .$$

Ruijsenaars proved that $w_m = \phi_m$ are **angle variables** and $\lambda_m = I_m$ are **action variables**.

Convenient measure of random ensemble

Consider L_{kr} as a random matrix depending on p and q with the **natural** measure

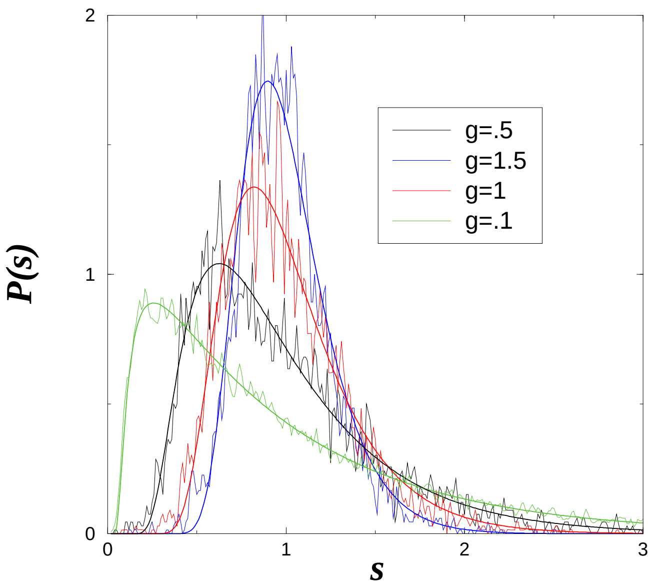
$$
\begin{aligned}
\mathrm{d}L &\sim \exp\left[-\alpha \mathrm{Tr} L^\dagger L - \beta \sum_k q_k^2\right] \mathrm{d}p\mathrm{d}q \\
&\equiv \exp\left[-\alpha \left(\sum_k p_k^2 + g^2 \sum_{i \neq j} \frac{1}{(q_i - q_j)^2}\right) - \beta \sum_k q_k^2\right] \mathrm{d}p\mathrm{d}q
\end{aligned}
$$

In variables λ and w this distribution can be rewritten as

$$
\begin{aligned}
\mathrm{d}L &\sim \exp\left[-\alpha \sum_m \lambda_m^2 - \beta \mathrm{Tr} Q^\dagger Q\right] \mathrm{d}\lambda\mathrm{d}w \\
&\equiv \exp\left[-\alpha \sum_m \lambda_m^2 - \beta \left(\sum_m w_m^2 + g^2 \sum_{m \neq n} \frac{1}{(\lambda_m - \lambda_n)^2}\right)\right] \mathrm{d}\lambda\mathrm{d}w\ .
\end{aligned}
$$

The integration over w gives a constant eigenvalues of the random matrix L are distributed as

$$P(\lambda_1, \ldots, \lambda_N) \sim \exp\left[-\alpha \sum_m \lambda_m^2 - \beta g^2 \sum_{m \neq n} \frac{1}{(\lambda_m - \lambda_n)^2}\right] .$$



$$\mathbf{p(s) = a_0 e^{-a_1 s - g^2/s^2}}$$

Angle-action variables for Ruijsenaars model

$$\mathbf{L_{kp}} = \mathrm{e}^{-\mathrm{i}\pi(\mathbf{N-1})\mathbf{a}} \mathrm{e}^{\mathrm{i}\boldsymbol{\Phi}_\mathbf{k} + \mathrm{i}(\mathbf{q_k} - \mathbf{q_r})/\mathbf{2}} \mathbf{C_{kp}}(\mathbf{a}, \mathbf{q})$$

$$\mathbf{C_{kp}}(\mathbf{a}, \mathbf{q}) = V_k^{1/2}(a; q) \frac{\sin \pi \mathbf{a}}{\sin \left(\frac{\mathbf{q_k} - \mathbf{q_p}}{2} + \pi \mathbf{a} \right)} V_p^{1/2}(-a; q) \ .$$

$$V_j(a, q) = \prod_{s \neq j} \frac{\sin \left(\frac{\mathbf{q_j} - \mathbf{q_s}}{2} + \pi \mathbf{a} \right)}{\sin \left(\frac{\mathbf{q_j} - \mathbf{q_s}}{2} \right)} \ ,$$

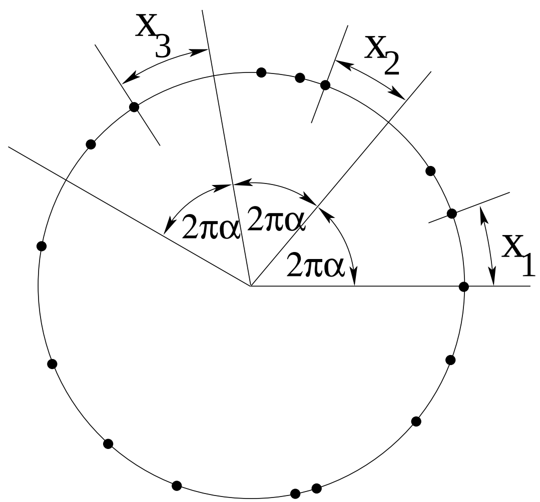
$$\sum_{p=1}^{N} L_{kp} u_p(\gamma) = \lambda_\gamma u_k(\gamma)$$

$$Q_{\gamma\xi} = \sum_{n=1}^{N} u_n(\gamma) \mathrm{e}^{\mathrm{i} q_n} u_n^*(\xi) \ .$$

$$Q_{\gamma\xi} = \mathrm{e}^{\mathrm{i}\pi(N-1)a} \mathrm{e}^{\mathrm{i}\phi_\gamma + \mathrm{i}(\lambda_\gamma - \lambda_\xi)/2} C_{\lambda\xi}(-a, \lambda) \ .$$

Unitarity condition

L_{kp} is unitary **iff** all $V_j(a;q)V_j(-a,q)$ are positive. When this condition is satisfied $Q_{\gamma\xi}$ is also unitary.



Let us divide the unit circle in anticlock-wise direction into sectors of angle $2\pi\alpha$ and denote the numbers of eigenvalues inside the k^{th} sector by n_k.

$n_2 = n_1 + 1$. For $k \geq 3$ $\mathbf{n_k = n_1 + \Theta(x_{k+1} - x_k)}$

The sum of eigenvalues numbers over all K sectors obey the following inequalities

$$N - n_1 - 1 \leq \sum_{k=1}^{K} n_k \leq N - 1 \ .$$

$$K n_1 + 1 \leq \sum_{k=1}^{K} n_k \leq K(n_1 + 1) \ .$$

$$\frac{N}{K+1} - 1 \leq n_1 \leq \frac{N-2}{K} \ .$$

$$\frac{N}{a} - 1 < K < \frac{N}{a}$$

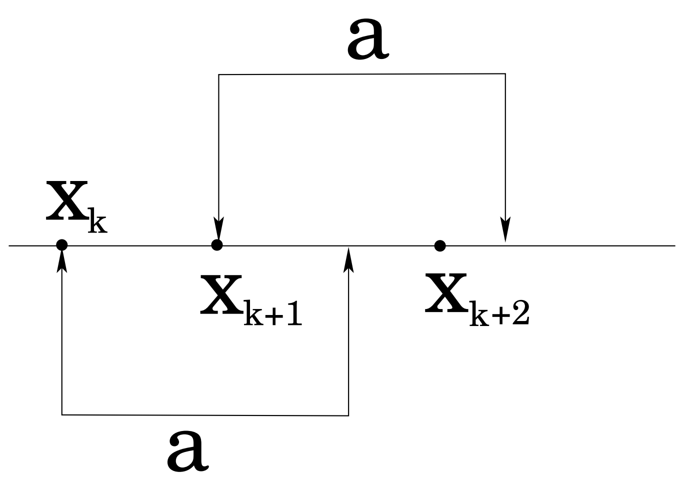
$$a - 1 - \frac{a^2}{N+a} < n_1 < a + \frac{a(a-2)}{N-a} \ .$$

At sufficiently large N: $\mathbf{a - 1 < n_1 < a} \longrightarrow \mathbf{n = [a]}$

Lemma: *When $\alpha = a/N$ and $N > N_*$ at the angular distance of $2\pi a/N$ from each eigenvalue there exist exactly $[\mathbf{a}]$ other eigenvalues.*

Direct consequences of the Lemma.

- When $0 < a < 1$ the **minimal** distance between 2 eigenvalues is $2\pi a/N$.
- When $a > 1$ the **maximal** distance between 2 eigenvalues is $2\pi a/N$.



Mutual positions of eigenvalues for $1 < a < 2$.

Joint probability of eigenvalues

Initial probability of matrix elements

$$\mathrm{d}P(\vec{p},\vec{q}\,) = R(\vec{q}\,)\mathrm{d}\vec{p}\,\mathrm{d}\vec{q}$$

$R(\vec{q})$ = restrictions imposed on $\vec{q}$ that matrix L is unitary

Transforming to action-angle variables

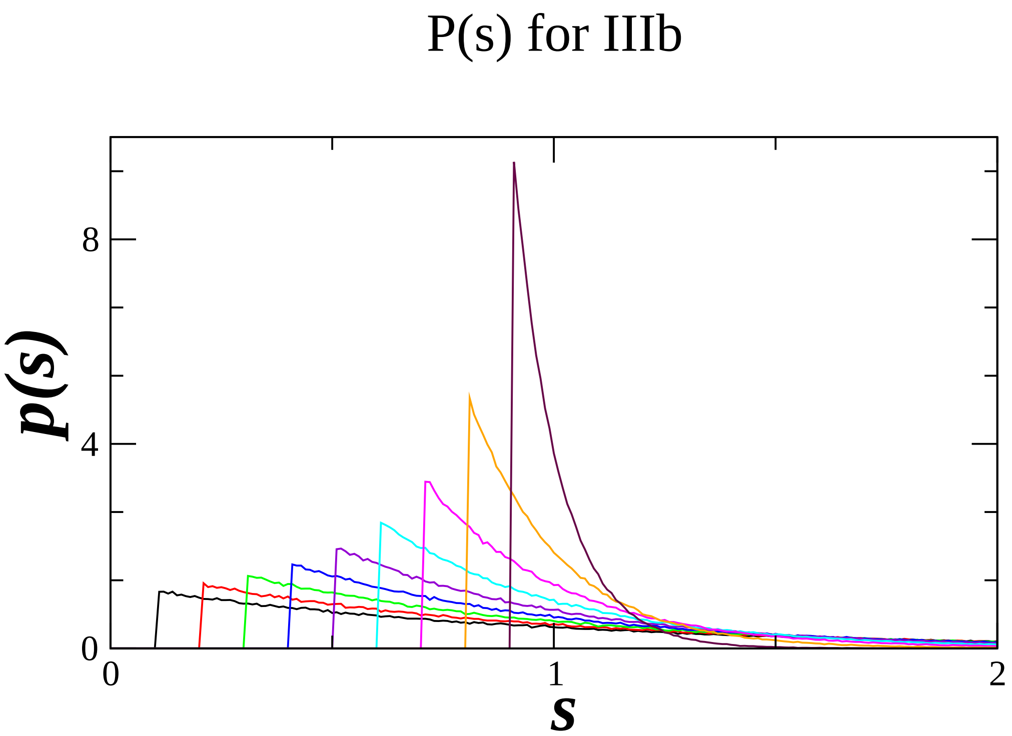
$$\mathrm{d}P(\vec{\lambda},\vec{\phi}\,) = R(\vec{\lambda}\,)\mathrm{d}\vec{\lambda}\,\mathrm{d}\vec{\phi}$$

Integrating over all phases

$$\mathrm{d}\mathbf{P}(\vec{\lambda}\,) = \mathbf{R}(\vec{\lambda}\,)\mathrm{d}\vec{\lambda}$$

$\mathbf{0 < a < 1}$

Poisson distribution shifted by a



$$\alpha = .1, .2, \ldots, .9.$$

Transfer operator for $\mathbf{1 < a < 2}$

$$x_1 < x_2 < \ldots < k_k < \ldots < x_N \ .$$

The restriction means that for all k

$$x_k + a < x_{k+2} < x_{k+1} + a \ .$$

Differences between consequtive eigenvalues $\xi_k = x_{k+1} - x_k$ obeys

$$0 < \xi_{k+1} < a \ , \ \ a < \xi_{k+1} + \xi_k \ .$$

Introduce 2 functions

$$f(x) = \begin{cases} 1 & \text{when } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{when } a < x \\ 0 & \text{otherwise} \end{cases} \ .$$

Joint probability density of $N + 1$ eigenvalues inside an interval of length L is given by the following expression

$$p(\xi_1, \xi_2, \ldots, \xi_N) = \frac{1}{Z_N(L)} \prod_{j=1}^{N} f(\xi_k) g(\xi_k + \xi_{k+1}) \delta \left(L - \sum_{k=1}^{N} \xi_k \right)$$

$Z_N(L)$ is the normalization constant

$$Z_N(L) = \int_0^\infty \mathrm{d}\xi_1 \ldots \int_0^\infty \mathrm{d}\xi_N \prod_{j=1}^{N} f(\xi_k) g(\xi_k + \xi_{k+1}) \delta \left(L - \sum_{k=1}^{N} \xi_k \right) \ .$$

transfer operator with interaction between 2 near-by points. Exactly as was done in B., Gerland, Schmit (1999).

Direct calculations

Calculate the Laplace tarnsform of $Z_N(L)$

$$g_N(t) = \int_0^\infty Z_N(L)\mathrm{d}L = \int_0^\infty \mathrm{d}\xi_1 \ldots \int_0^\infty \mathrm{d}\xi_N \prod_{j=1}^{N} \mathrm{e}^{-t\xi} f(\xi_k) g(\xi_k + \xi_{k+1}) \,.$$

It is the formal trace of a transfer operator

$$g_N(t) = \mathrm{Tr}K^N = \int_0^\infty \mathrm{d}\xi_1 \ldots \int_0^\infty \mathrm{d}\xi_N \, K(\xi_1, \xi_2)K(\xi_2, \xi_3) \ldots K(\xi_{N-1}, \xi_N)K(\xi_N, \xi_1)$$

$$K(\xi, \xi') = \sqrt{f(\xi)}\mathrm{e}^{-t\xi/2} g(\xi + \xi') \sqrt{f(\xi')}\mathrm{e}^{-t\xi'/2} \,.$$

It is real symmetric operator. Its eigenvalues, λ_j, and eigenfunctions, $\phi_j(\xi)$, are well defined

$$\int_0^\infty K(\xi, \xi')\phi_j(\xi')\mathrm{d}\xi' = \lambda_j \phi_j(\xi) \,.$$

The most important is the largest eigenvalue of the transfer operator, $\lambda_0(t)$, and the corresponding eigenfunctions, $\phi_0(t,\xi)$.

In the large N limit the dominant contribution to (**??**) is given by λ_0

$$g_N(t) \stackrel{N\to\infty}{\longrightarrow} [\lambda_0(t)]^N \ .$$

$Z_N(L)$ is computed from the inverse Laplace transformation

$$Z_N(L) = \frac{1}{2\pi \mathrm{i}} \int_{b-\mathrm{i}\infty}^{b+\mathrm{i}\infty} g_N(t) \mathrm{e}^{Lt} \mathrm{d}t \ .$$

In the saddle point approximation one

$$Z_N(L) \stackrel{N\to\infty}{\longrightarrow} [\lambda_0(c)]^N \mathrm{e}^{Lc}$$

c is the solution of the saddle point equation

$$\Delta + \frac{\lambda_0'(c)}{\lambda_0(c)} = 0, \ \Delta = L/N \text{ is mean level density.}$$

Correlation functions

$$p(\xi_1, \xi_2, \ldots, \xi_n) = \frac{1}{\lambda_0^{n-1}(c)} \phi_0(\xi_1) K(\xi_1, \xi_2) K(\xi_2, \xi_3), \ldots, K(\xi_{n-1}, \xi_n) \phi_0(\xi_n) \; .$$

$p(\xi)$ is the nearest-neighbor distribution

$$p(s) = (\phi_0(s))^2 \; .$$

$$p(n, s) = \int_0^\infty \mathrm{d}\xi_1 \ldots \int_0^\infty \mathrm{d}\xi_n p(\xi_1, \ldots, \xi_n) \delta\left(s - \sum_{j=1}^{n} \xi_j\right) \; .$$

Explicit calculations for $1 < a < 2$

Eigenfunctions are non-zero only when $0 < x < a$, and

$$e^{-t\xi/2} \int_{a-\xi}^{a} e^{-t\xi'/2} \phi(\xi') d\xi' = \lambda \phi(\xi) \; .$$

Let us look for the solution in the form

$$\phi(\xi) \sim \sinh \rho\xi$$

It is eigenfunctions provided

$$e^{2\rho a} = \frac{t/2 - \rho}{t/2 + \rho}$$

and the eigenvalue λ is

$$\lambda = \frac{e^{(\rho - t/2)a}}{\rho - t/2} \; .$$

When $1 < a < 2$ the nearest-neighbor distribution is

$$p(s) = \begin{cases} A\sinh^2(\rho s) & \text{when } 1 < a < 4/3 \\ \frac{81}{64}s^2 & \text{when } a = 4/3 \\ A\sin^2(\rho s) & \text{when } 4/3 < a < 2 \end{cases}$$

A and ρ from the normalization conditions:

$$\int_0^a p(s)\mathrm{d}s = 1 \ , \quad \int_0^a sp(s)\mathrm{d}s = 1 \ .$$

The next-nearest distribution ($\mu(\rho) = \rho\cos(\rho a)/\sin(\rho a)$):

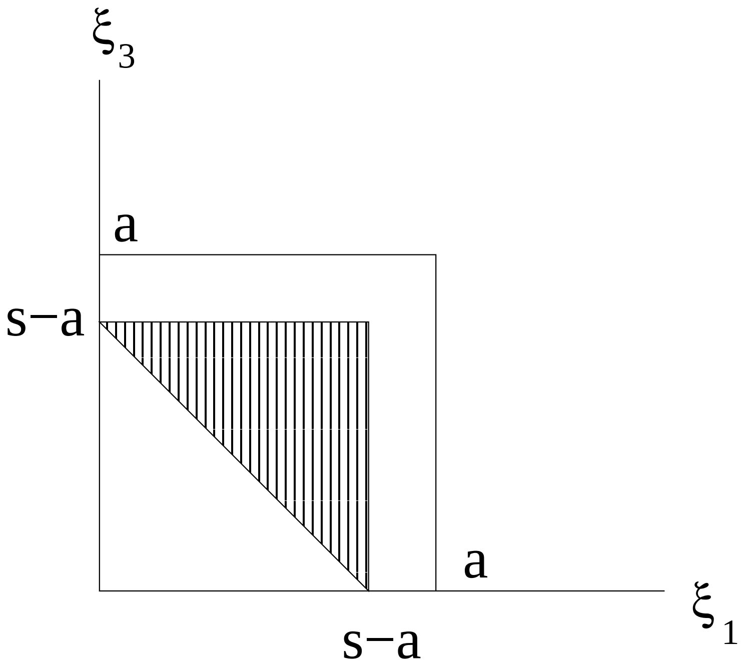
$$p(2,s) = A_2 \mathrm{e}^{\mu(\rho)s} \int_{s-a}^{a} \sin(\rho y)\sin(\rho(s-y))\mathrm{d}y \ .$$

In particular, for $a = 4/3$

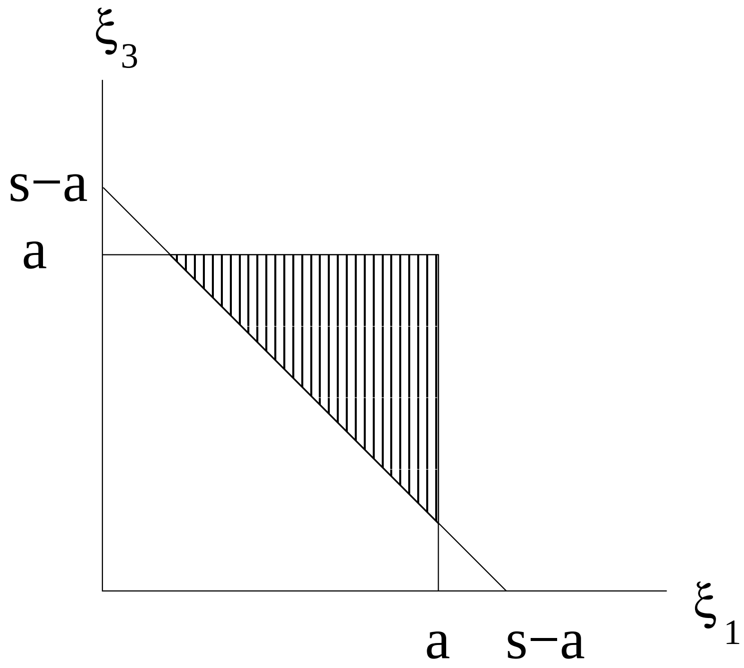
$$p(2,s) = (-\frac{3}{2} + \frac{27}{16}s - \frac{81}{512}s^3)\mathrm{e}^{3s/4-1} \ .$$

$p(3, s)$ is non-zero when $a < s < 3a$ and

$$p(3, s) = A_3 \mathrm{e}^{2\mu(\rho)s} \int_{\mathcal{D}(s)} \sin \rho\xi_1 \mathrm{e}^{-\mu(\rho)(\xi_1+\xi_3)} \sin \rho\xi_3 \;\mathrm{d}\xi_1 \;\mathrm{d}\xi_3$$



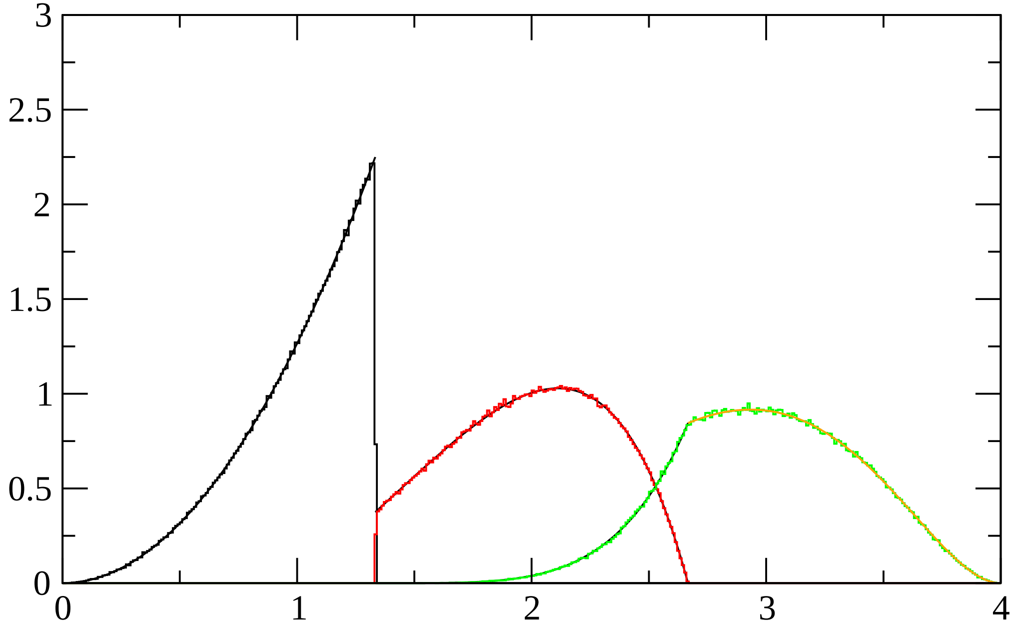
$a < s < 2a$



$2a < s < 3a$

For $a = 4/3$ we obtain

$$p(3,s) = \begin{cases} (\frac{3}{4} - \frac{81}{32}s + \frac{81}{512}s^3)\mathrm{e}^{3s/4-1} + \frac{81}{64}s^2 & \text{when } 4/3 < s < 8/3 \\ (-\frac{9}{4} + \frac{27}{32}s - \frac{81}{512}s^3)\mathrm{e}^{3s/4-1} + 9\mathrm{e}^{3s/2-4} & \text{when } 8/3 < s < 4 \end{cases} .$$



General case $n < a < n+1$

In this case inside the interval of length a from an eigenvalue there exit exactly n other eigenvalues.

$$0 < \xi_k + \xi_{k+1} + \ldots + \xi_{k+n-1} < a$$
$$a < \xi_k + \xi_{k+1} + \ldots + \xi_{k+n} \ .$$

Joint probability of eigenvalues

$$p(\xi_1, \xi_2, \ldots, \xi_N) = \frac{1}{Z_N(L)} \prod_{j=1}^{N} f(\xi_k + \ldots + \xi_{k+n-1}) g(\xi_k + \ldots + \xi_{k+n})$$
$$\times \delta\left(L - \sum_{k=1}^{N} \xi_k\right)$$

Interaction only between n nearest-neighbor eigenvalues.

The transfer operator in this case depends on 2 sets of variables $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ and $\vec{\xi'} = (\xi'_1, \ldots, \xi'_n)$ shifted by one unit i.e. $\xi_2 = \xi'_1$, $\xi_3 = \xi'_2$, …, $\xi_n = \xi'_{n-1}$.

The explicit form of the transfer operator is the following

$$K(\vec{\xi}, \vec{\xi'}) = \delta(\xi_2 - \xi'_1) \ldots \delta(\xi_n - \xi'_{n-1})$$
$$\times e^{-t\xi_1/2} \sqrt{f(\xi_1 + \ldots + \xi_n)} g(\xi_1 + \ldots + \xi_n + \xi'_n) \sqrt{f(\xi'_1 + \ldots + \xi'_n)} e^{-t\xi'_n/2} \ .$$

The eigenvalue eqution

$$\int K(\vec{\xi}, \vec{\xi'}\,) \phi(\vec{\xi'}\,) \mathrm{d}\vec{\xi'} = \lambda \phi(\vec{\xi}\,)$$

reduces to a one-dimensional equation

$$e^{-t\xi_1/2} \int_0^\infty e^{-tz/2} g(\xi_1 + \xi_2 + \ldots + \xi_n + z) \phi(t; \xi_2, \ldots, \xi_n, z) \mathrm{d}z = \lambda(t) \phi(t; \xi_1, \ldots, \xi_n) \ .$$

The probability of k consecutive spasings has a different form for $k \leq n$ and $k > n$. In the former case

$$p(\xi_1, \ldots, \xi_k) = \int_0^a \mathrm{d}\xi_{k+1} \ldots \int_0^a \mathrm{d}\xi_n \phi_0(c; \xi_1, \ldots, \xi_n)\phi_0(c; \xi_n, \ldots, \xi_1),$$

and in the latter case

$$p(\xi_1, \ldots, \xi_k) = \lambda_0(c)^{-k+n}\phi_0(c; \xi_k, \ldots, \xi_{k-n})\phi_0(c; \xi_1, \ldots, \xi_n)\mathrm{e}^{-c\sum_{s=1}^{k}\xi_s}$$
$$\times \prod_{j=1}^{k-n+1} f(\xi_j + \ldots + \xi_{j+n-1}) \prod_{j=1}^{k-n} g(\xi_j + \ldots + \xi_{j+n})$$

$\mathbf{2 < a < 3}$

The eigenvalue equation takes the form

$$\mathrm{e}^{\mu\xi_1}\int_{a-\xi_1-\xi_2}^{a-\xi_2}\mathrm{e}^{\mu\xi_3}\phi(\xi_2,\xi_3)\mathrm{d}\xi_3=\lambda\phi(\xi_1,\xi_2)\ .$$

Solution of this equation in the form similar to Bethe anzatz

$$\begin{aligned}\phi(\xi_1,\xi_2) &= \mathrm{e}^{\alpha\xi_1+\beta\xi_2}+\mathrm{e}^{-\beta\xi_1+(\alpha-\beta-\mu)\xi_2}+\mathrm{e}^{(-\alpha+\beta+\mu)\xi_1-\alpha\xi_2}\\ &\quad -\mathrm{e}^{-\beta\xi_1-\alpha\xi_2}-\mathrm{e}^{\alpha\xi_1+(\alpha-\beta-\mu)\xi_2}-\mathrm{e}^{(-\alpha+\beta+\mu)\xi_1+\beta\xi_2}\ .\end{aligned}$$

This function is a solution provided

$$\frac{\mathrm{e}^{a(\mu+\beta)}}{\mu+\beta}=\frac{\mathrm{e}^{a(\mu-\alpha)}}{\mu-\alpha}=\frac{\mathrm{e}^{a(\alpha-\beta)}}{\alpha-\beta}=-\lambda\ .$$

$$\alpha a=\frac{1}{2}x_1+\mathrm{i}x_2\ ,\ \ \beta a=-\frac{1}{2}x_1+\mathrm{i}x_2\ ,\mu a=\frac{1}{2}x_1+x_3$$

with real x_1, x_2, and x_3.

Under this substitution the eigenfunction becomes

$$\phi(\xi_1, \xi_2) = \mathrm{e}\gamma(\xi_1 - \xi_2) \left(\sin(\delta(\xi_1 + \xi_2)) - \mathrm{e}^{\rho\xi_2} \sin(\delta(\xi_1)) - \mathrm{e}^{-\rho\xi_1} \sin(\delta(\xi_2)\right)$$

where $\gamma = x_1/(2a)$, $\delta = x_2/a$, $\rho = (x_1 - x_3)/a$.

$$\frac{\mathrm{e}^{x_1}}{x_1} = \frac{\mathrm{e}^{x_3 + \mathrm{i}x_2}}{x_3 + \mathrm{i}x_2} = \frac{\mathrm{e}^{x_3 - \mathrm{i}x_2}}{x_3 - \mathrm{i}x_2} = -\frac{\lambda}{a} \ .$$

It follows that $x_3 = \frac{x_2}{\tan(x_2)}$ and

$$\frac{\mathrm{e}^{x_1}}{x_1} = \frac{\sin(x_2)}{x_2} \mathrm{e}^{\frac{x_2}{\tan(x_2)}} \ .$$

The nomalization condition is

$$a = \frac{1}{1 - 1/x_1} + \frac{2 - \sin(2x_2)/x_2}{1 + \sin^2(x_2)/x_2^2 - \sin(2x_2)/x_2} \ .$$

The largest eigenvalue corresponds to $x_1 < 0$, $\pi < x_2 < 2\pi$.

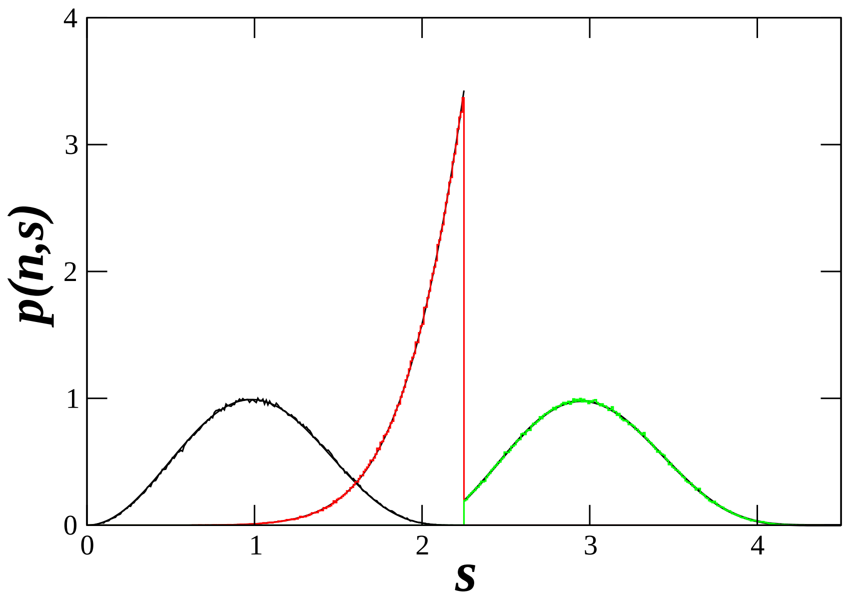
The knowledge of these parameters permit to calculate the eigenfunction from which all nearest-neighbor distribution are expressed:

$$p(1,s) = A\int_0^{a-s} \phi(s,y)\phi(y,s)\mathrm{d}y \ ,$$

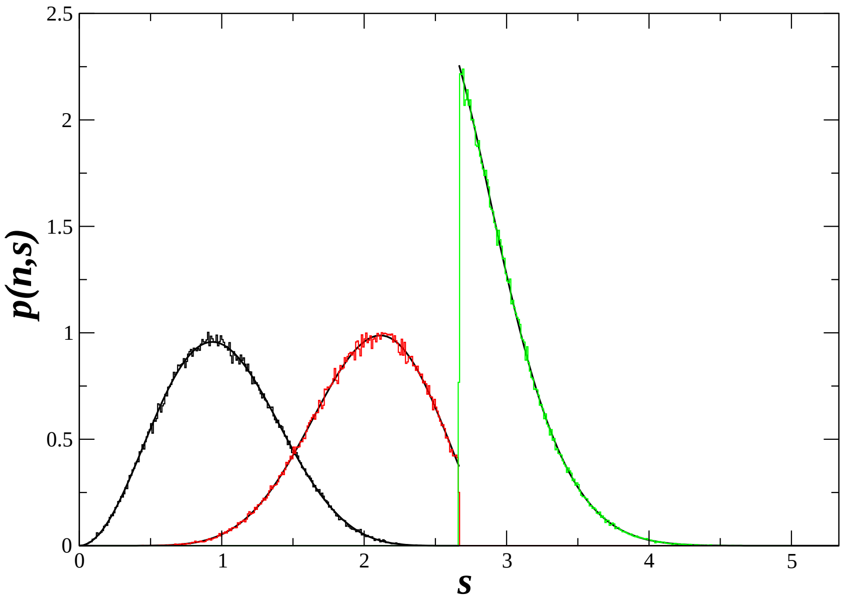
$$p(2,s) = A\int_0^{s} \phi(s-y,y)\phi(y,s-y)\mathrm{d}y \ ,$$

and

$$p(3,s) = \frac{A}{\lambda}\int_{s-a}^{a} \mathrm{d}x\mathrm{e}^{\mu x}\int_{s-a}^{s-x} \mathrm{d}z\mathrm{e}^{\mu z}\phi(s-x-z,x)\phi(s-x-z,z)\mathrm{d}y \ .$$



Three first nearest-neighbor distributions for $a = 9/4$. $N = 701$ and 1000 iterations. Red, blue, and blue lines are theoretical predictions.



Three first nearest-neighbor distributions for $a = 8/3$. $N = 701$ and 1000 iterations.

Summary

- Lax matrices of integrable classical systems give ensembles of random matrices with intermediate statistics.
- For these ensembles it is possible to calculate exactly the joint probability of eigenvalues though all ensembles have no rotational symmetry.
- For a large number of ensembles correlation functions of eigenvalues can be computed analytically.
- For all cases eigenfunctions have fractal properties.
- The quantization of the pseudo-integrable map corresponds to Ruijsenaars model.
- New perspectives for intermediate statistics.