The Asymmetric Simple Exclusion Process: Integrable Structure and Limit Theorems

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Joint work with Harold Widom

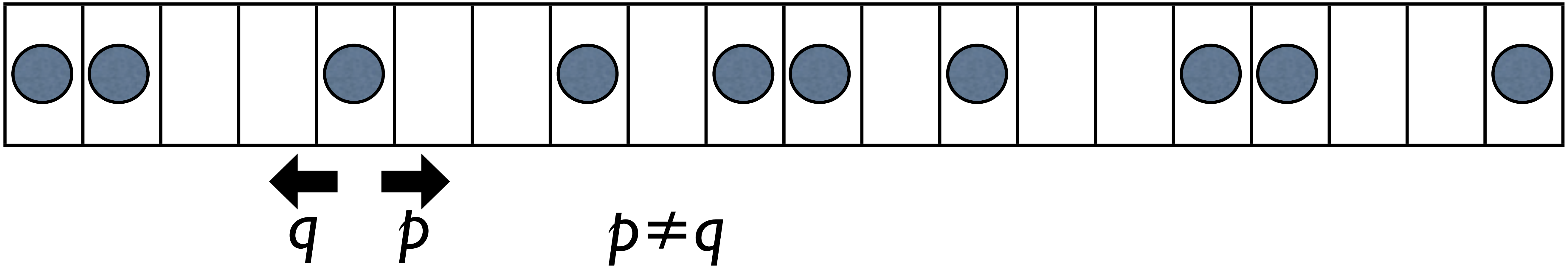
The asymmetric simple exclusion process (ASEP): Introduced in 1970 by Frank Spitzer in "Interaction of Markov Processes."

The "default stochastic model for transport phenomena". The ``Ising model of nonequilibrium phenomena".

ASEP is a model for interacting particles on a lattice.

Frank Spitzer

ASEP on Integer Lattice



$p \neq q$

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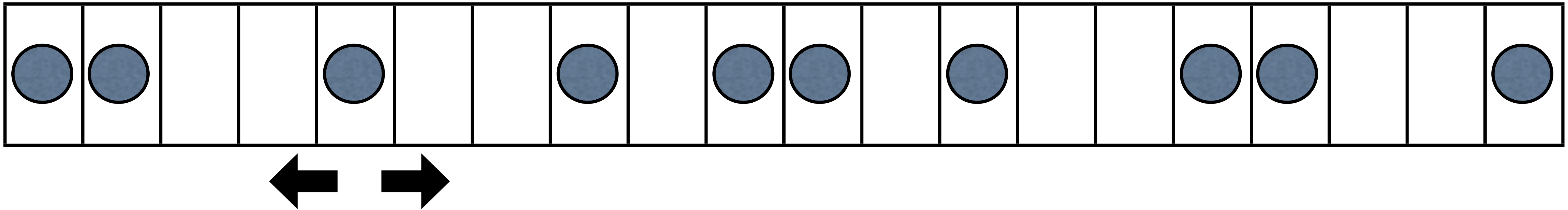

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- Each particle has an alarm clock -- exponential distribution with parameter one

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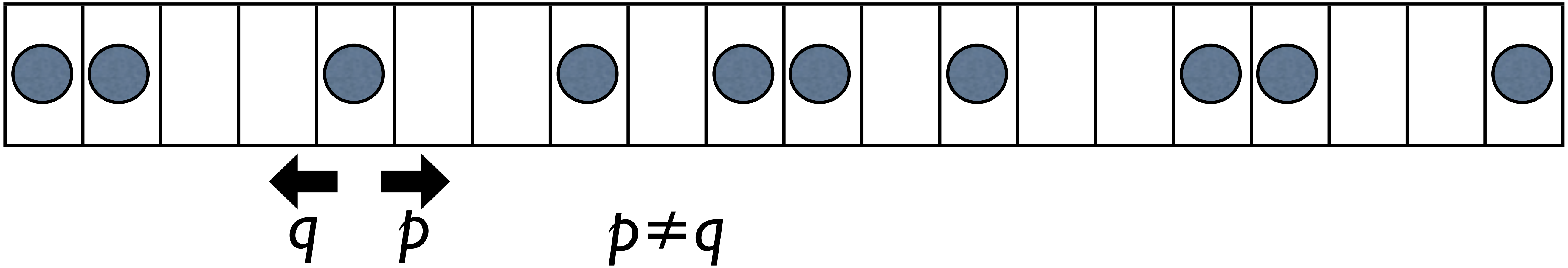

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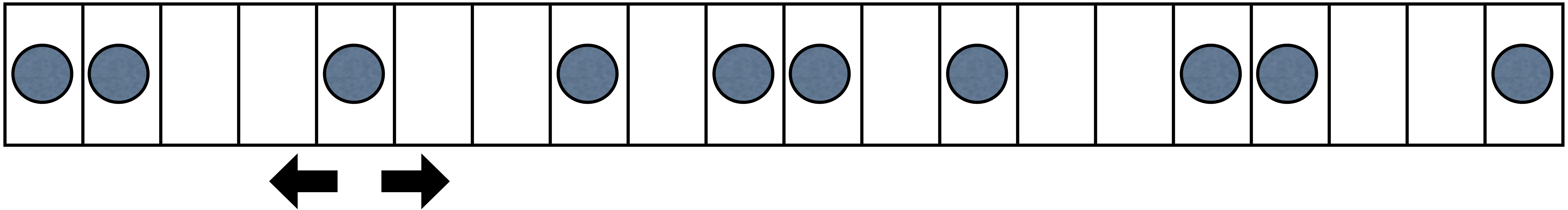
- When alarm rings particle jumps to right with probability p and to the left with probability q

ASEP on Integer Lattice



q p $p \neq q$

- Each particle has an alarm clock -- exponential distribution with parameter one
- When alarm rings particle jumps to right with probability p and to the left with probability q
- Jumps are suppressed if neighbor is occupied

Initial Conditions

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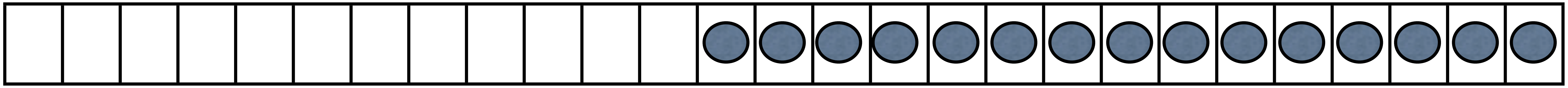

Step Initial Condition, q>p

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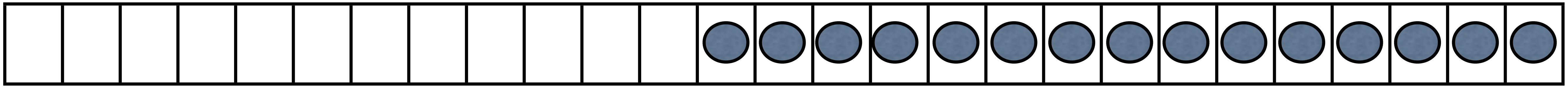

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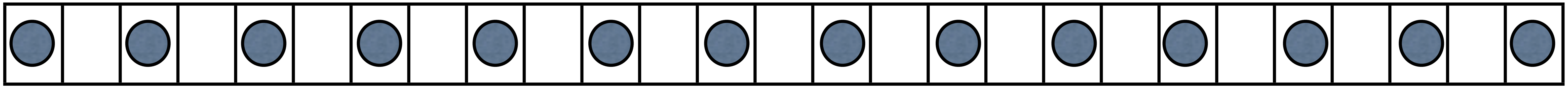

Flat Initial Condition

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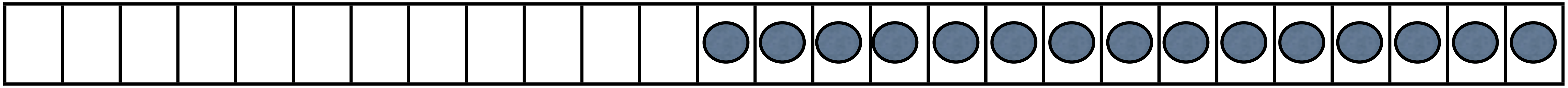

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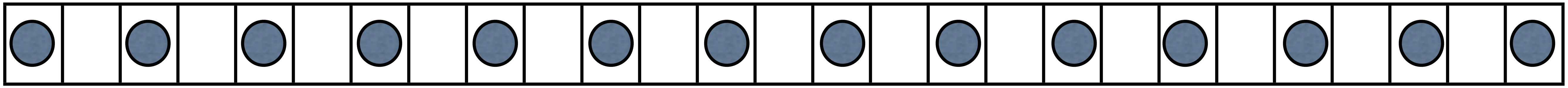

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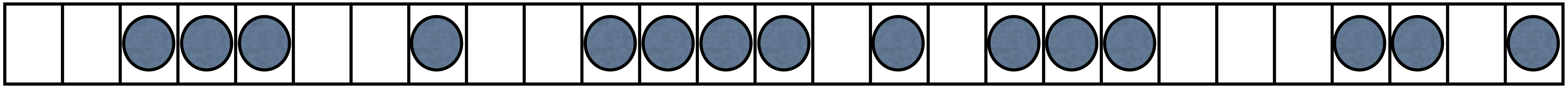
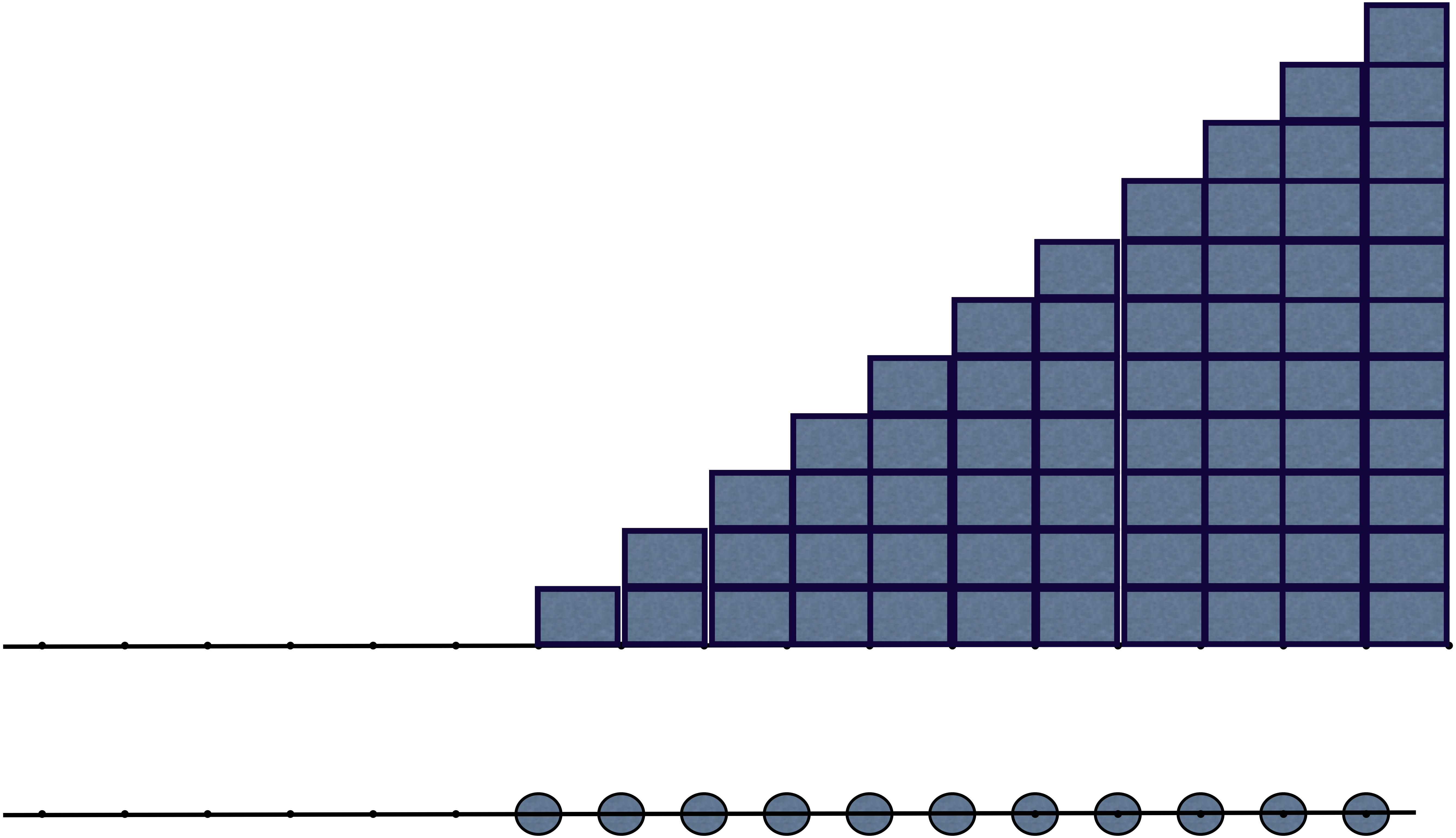


Random: Product Bernoulli measure

Growth Processes & ASEP



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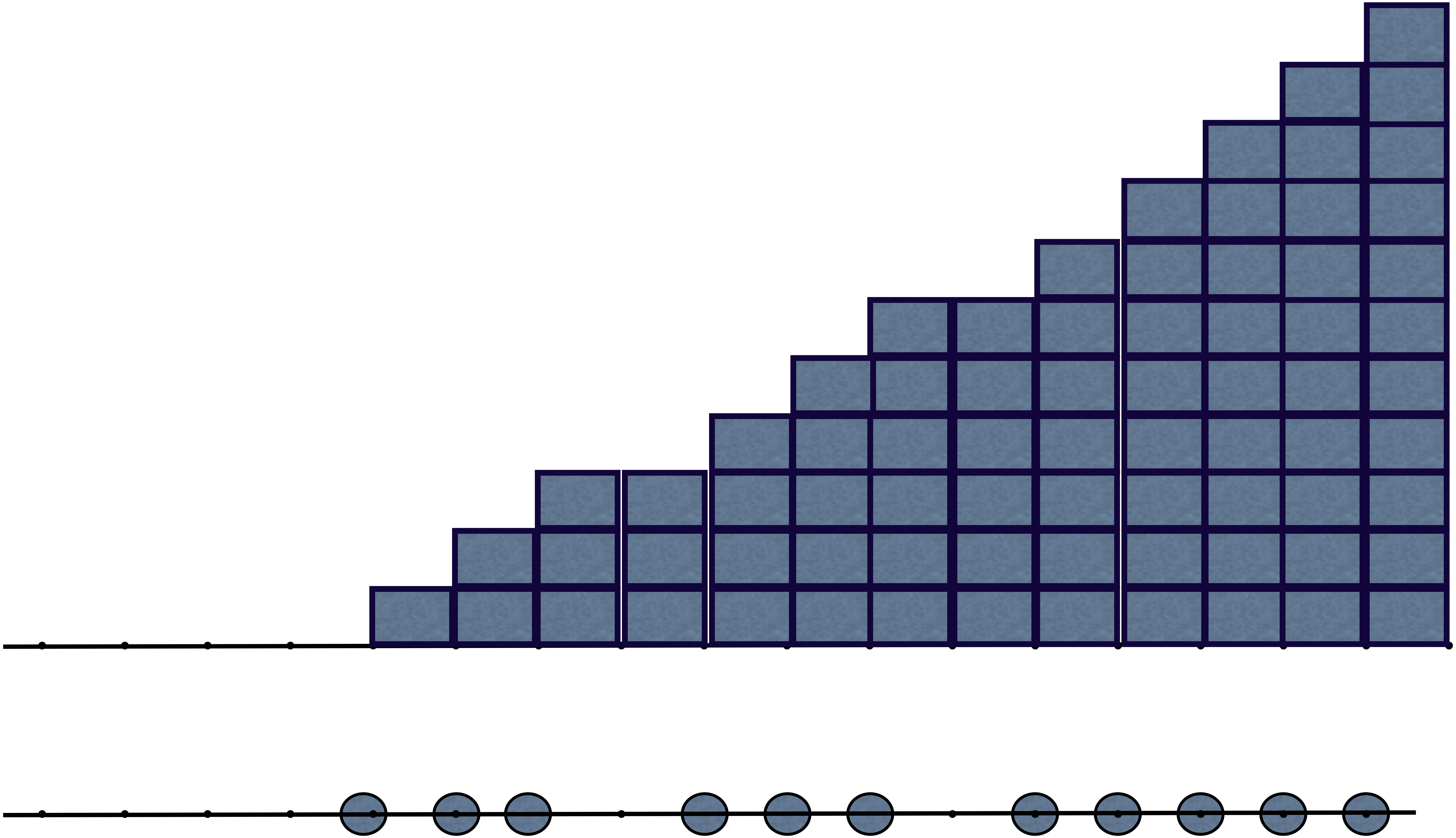

KPZ Equation & Growth Processes

→ Kardar, Parisi & Zhang

$$\frac{\partial h}{\partial t} = \underset{\substack{\uparrow \\ \text{diffusion}}}{\nu \frac{\partial^2 h}{\partial x^2}} + \underset{\substack{\uparrow \\ \text{growth}}}{\lambda \left(\frac{\partial h}{\partial x}\right)^2} + \underset{\substack{\uparrow \\ \text{noise}}}{w}$$

$$u(x,t) = \frac{\partial h}{\partial x} \longrightarrow \text{Noisy Burgers eqn}$$

Remarks

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- Physicists expect KPZ equation to describe a large class of stochastically growing interfaces: KPZ Universality
- KPZ eqn mathematically difficult to handle so make discrete approximation in space
- ASEP is one discrete version of KPZ; thus expect ``universal behavior" in limit theorems
- TASEP is ASEP with jumps only to left or jumps only to right. TASEP is a simpler model (determinantal process)

Total Current I(x,t)

Step Initial Condition

Take q>p net drift to left

$I(x,t)$ = # of particles $\leq x$ at time t, $x \leq 0$

Let $\eta(x,t)=1$ if particle at x at time t

otherwise 0

so $I(x,t)=\Sigma_{y\leq x}\,\eta(y,t)$

Current & Position of mth particle

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$x_m(t)$=position of m^{th} particle from left at time t, $x_m(0)=m$

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Event:

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From this and exclusion property:

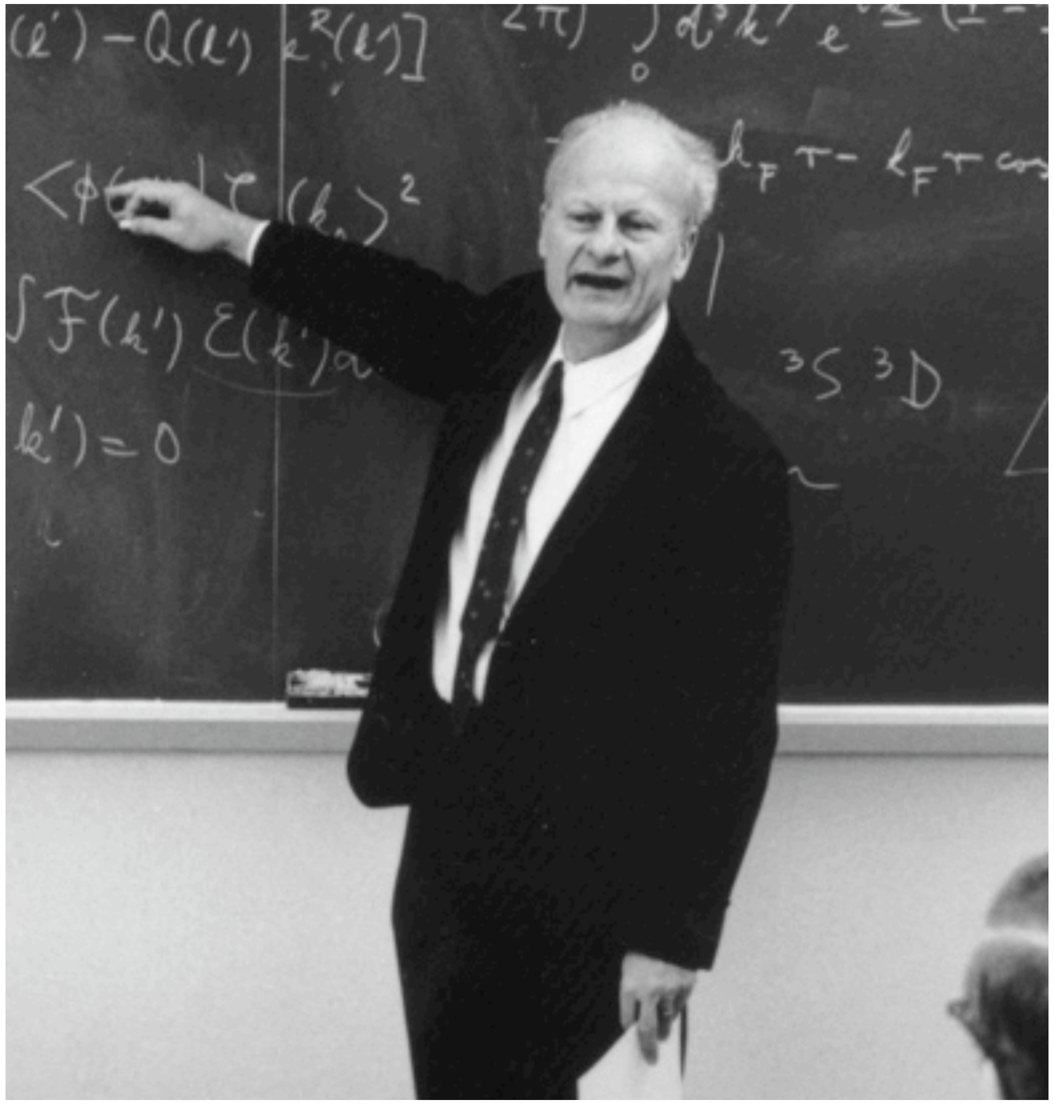
$$\mathrm{Prob}(I(x,t)\leq m) = 1-\mathrm{Prob}(x_{m+1}(t)\leq x)$$

Integrable Structure of ASEP

We solve the Kolmogorov forward equation ("master equation") for the transition probability Y→X:

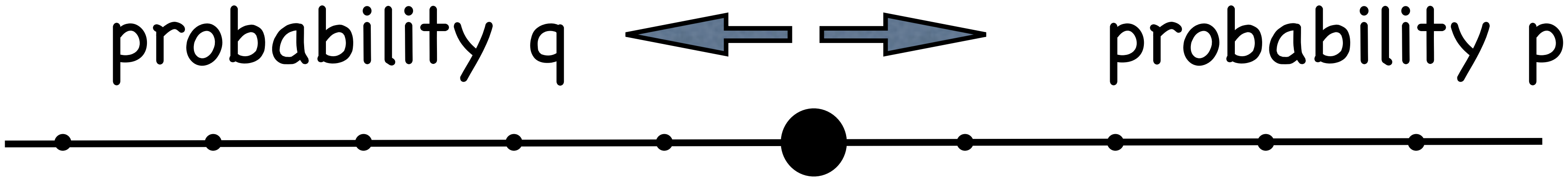
$$P_Y(X;t)$$

Main idea comes from the Bethe Ansatz (1931)



Hans Bethe in 1967

N=1 ASEP



probability q

probability p

Let P$_Y$(X;t)=probability Y→X at time t.

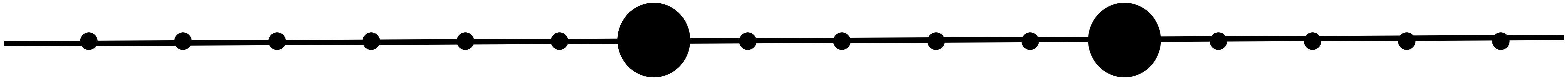
Master equation:

$$\frac{dP}{dt} = p\,P(x-1;t) + q\,P(x+1;t) - P(x;t)$$

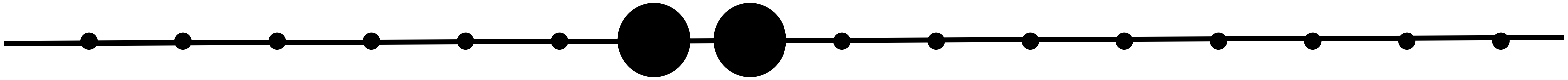
$$P_y(x;t) = \int_{\mathcal{C}} \xi^{x-y-1} e^{t\varepsilon(\xi)}\,d\xi$$

$$\varepsilon(\xi) = \frac{p}{\xi} + q\,\xi - 1$$

N=2 ASEP



Master equation takes simple form for this configuration



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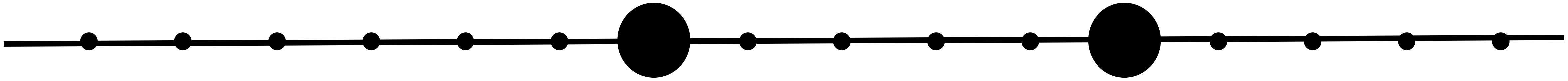

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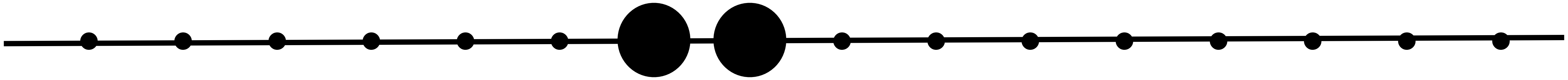
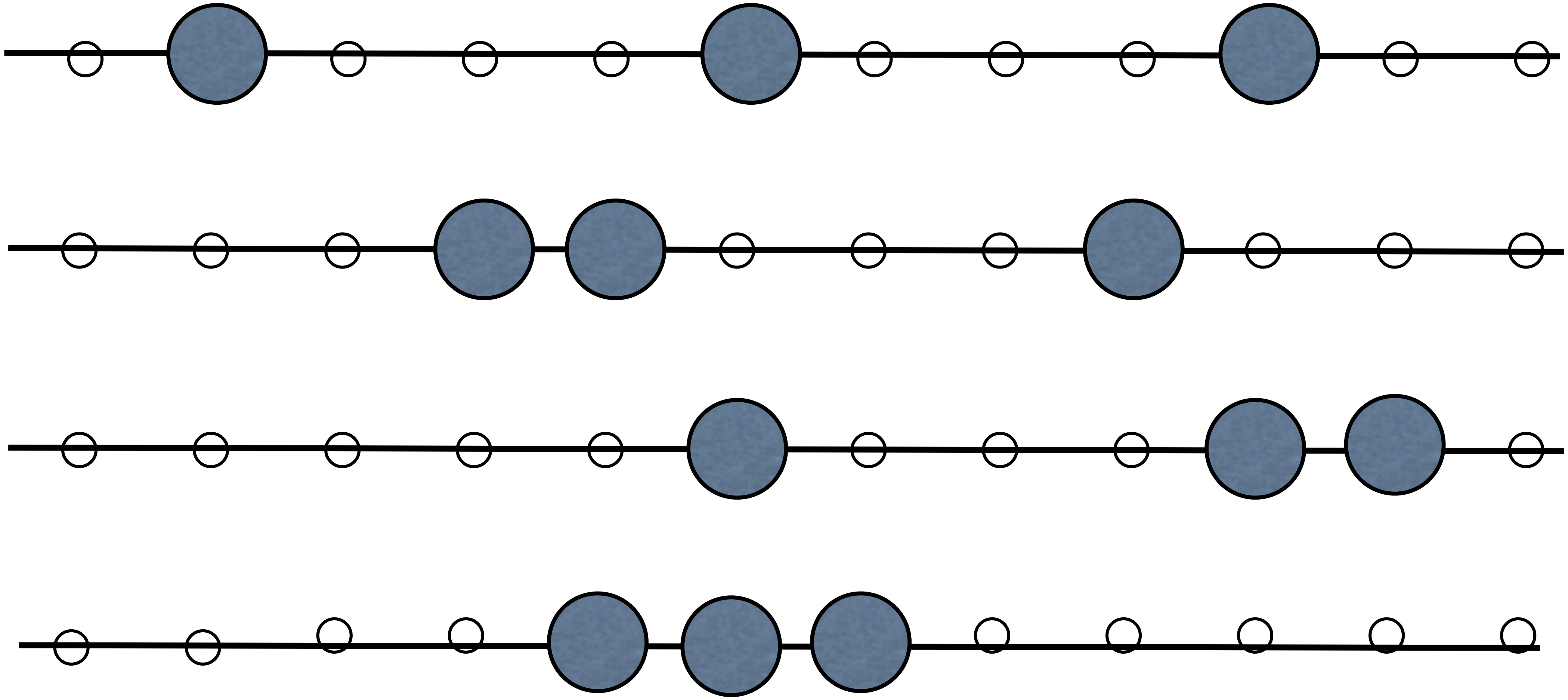


Master equation reflects exclusion for this configuration

→ Impose boundary conditions for first equation so that if satisfied the second equation is automatically satisfied --- Bethe's Idea

Important Point

New boundary conditions arise for N=3, 4,...



Last configuration requires new BC --
automatically satisfied by 2-particle BC

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For any $\xi_1,\ldots,\xi_N \in \mathbf{C}\backslash\{0\}$ and any permutation σ a solution is $$\prod_j \xi_{\sigma(j)}^{x_j}\, e^{t\varepsilon(\xi_j)}$$

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$$\prod_j \xi_{\sigma(j)}^{x_j}\, e^{t\varepsilon(\xi_j)}$$

Can take linear combination or integral of a linear combination & have a solution:

$$\int \sum_{\sigma\in\mathcal{S}_N} F_\sigma(\xi) \prod_j \xi_{\sigma(j)}^{x_j} \prod_j e^{t\varepsilon(\xi_j)}\, d^N\xi$$

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- Gives condition on coefficients with result (this part same as the Yang & Yang analysis of XXZ spin Hamiltonian). If

$$A_\sigma = \operatorname{sgn}(\sigma)\,\frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i\xi_j - \xi_i)}$$

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then solution to DE that satisfies BC is

$$\sum_\sigma \int A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{t\varepsilon(\xi_i)}\, d^N\xi$$

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- To recap: We have found a solution to the master equation satisfying the BCs.
- Must show the solution at $X=\{x_1,...x_N\}$ & $t=0$ reduces to $\delta_{X,Y}$ where $Y=\{y_1,...,y_N\}$ is the initial configuration of N particles.
- Term for $\sigma=\mathrm{id}$ satisfies initial condition. Must show remaining $N!-1$ terms give zero at $t=0$. This is the new part of the problem.
- Have not yet specified the contours.

Theorem (TW): If $p \neq 0$ and r is small enough then

$$\mathbb{P}_Y(X;t) = \sum_{\sigma \in \mathcal{S}_N} \int_{\mathcal{C}_r^N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i} \prod_i \left(\xi_i^{-y_i - 1} \, e^{\varepsilon(\xi_i)\, t} \right) d^N \xi.$$

where

$$A_\sigma = \operatorname{sgn} \sigma \, \frac{\prod_{i<j} (p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)})}{\prod_{i<j} (p + q\xi_i \xi_j - \xi_i)}$$

and satisfies

$$\mathbb{P}_Y(X;0) = \delta_Y(X).$$

Remarks:

- There is no Ansatz in our work!
- Usual Bethe Ansatz calculates the spectrum of the operator. This leads to transcendental equations for the eigenvalues and issues of completeness of the eigenfunctions.
- We compute the semigroup directly. No spectral theory.

Marginal Distributions

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Case m=1:

Fix $x_1=x$, sum $P_Y(X;t)$ over allowed x_2, x_3, x_4,...

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Result is an expression involving N! terms. Use first miraculous identity to reduce sum to one term!

Here's the identity:

First Identity

$$\sum_{\sigma \in \mathcal{S}_N} \operatorname{sgn}\sigma \left(\prod_{i<j} (p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - \xi_{\sigma(i)}) \right.$$

$$\left. \times \frac{\xi_{\sigma(2)}\xi_{\sigma(3)}^2 \cdots \xi_{\sigma(N)}^{N-1}}{(1-\xi_{\sigma(2)}\xi_{\sigma(3)}\cdots\xi_{\sigma(N)})(1-\xi_{\sigma(3)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N)})} \right)$$

$$= p^{N(N-1)/2} \, \frac{(1-\xi_1\cdots\xi_N)\prod_{i<j}(\xi_j-\xi_i)}{\prod_i (1-\xi_i)}$$

- Using this identity we get for m=1 an expression for P(x₁(t)≤x) as a single N-dimensional integral with a product integrand. This expression is for finite-N ASEP

$$I(x,Y,\xi) = \prod_{i<j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \frac{1 - \xi_1 \cdots \xi_N}{(1-\xi_1)\cdots(1-\xi_N)} \prod_i \left(\xi_i^{x-y_i-1} e^{\varepsilon(\xi_i)t} \right)$$

Prob(x_1(t)=x) =

$$p^{N(N-1)/2}\int_{\mathcal{C}_r}\cdots\int_{\mathcal{C}_r} I(x,Y,\xi)\,d\xi_1\cdots d\xi_N$$

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- Sum of N! integrals reduced to one integral
- Form is not so useful to take N→∞
- We now expand contour outwards -- only residues that contribute come from ξ_i=1.
- Can take N→∞ in resulting expression to obtain

$$\sigma(S) : \quad = \quad \sum_{i \in S} i$$

$$\mathbb{P}(x_1(t) = x) \quad = \quad \sum_{S} \frac{p^{\sigma(S)-|S|}}{q^{\sigma(S)-|S|(|S|+1)/2}} \times$$

$$\int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} I(x, Y_S, \xi)\, d^{|S|}\xi$$

The sum is over all nonempty subsets of $\mathbb{Z}$

When p=0 only one term is nonzero, S={1}.

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- This involves finding a new identity

Second Identity

S ranges over subsets of $\{1, 2, \ldots, N\}$

$$\sum_{|S|=m} \prod_{i\in S,\, j\in S^c} \frac{p + q\xi_i\xi_j - \xi_i}{\xi_j - \xi_i} \cdot (1 - \prod_{j\in S^c} \xi_j)$$

$$= q^m \begin{bmatrix} N \\ m \end{bmatrix} (1 - \prod_{j=1}^{N} \xi_j).$$

$$[N] = \frac{p^N - q^N}{p-q}, \qquad [N]! = [N]\,[N-1]\,\cdots\,[1],$$

$$\begin{bmatrix} N \\ m \end{bmatrix} = \frac{[N]!}{[m]!\,[N-m]!}, \qquad (q - \text{binomial coefficient}),$$

Final series result for case $Y = \mathbb{Z}^+$

$$\begin{aligned}\mathbb{P}\left(x_m(t) \le x\right) \quad &= \quad (-1)^m \sum_{k \ge m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix}_\tau p^{(k-m)(k-m+1)/2} q^{km+(k-m)(k+m-1)/2} \\ &\times \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \prod_{i \ne j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{1}{(1-\xi_i)(q\xi_i - p)} \\ &\times \prod_i \left(\xi_i^x e^{\varepsilon(\xi_i)t} \right) d\xi_1 \cdots d\xi_k\end{aligned}$$

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- For p=0 only k=m term is nonzero
- Recognize double product as a determinant whose entries are a kernel, i.e. $K(\xi_i,\xi_j)$
- Result can then be expressed as a contour integral whose integrand is a Fredholm determinant

Fredholm determinant

- Let K(x,y) be a kernel function
- Fredholm expansion of det(I-λK):

$$\frac{(-1)^n}{n!}\int\cdots\int \det\left(K(\xi_i,\xi_j)\right)_{1\le i,j\le n}\, d\xi_1\cdots d\xi_n =$$

$$\int_{\mathcal{C}} \det\left(I-\lambda K\right)\,\frac{d\lambda}{\lambda^{n+1}}$$

- Can then do sum over k (q-Binomial theorem):

Final expression for mth particle distribution fn. Step initial condition

Set $\gamma = q - p > 0$, $\tau = q/p$ and define an integral operator K on circle C_R:

$$K(\xi, \xi') = q\,\frac{\xi'^{x} e^{t\varepsilon(\xi')/\gamma}}{p + q\xi\xi' - \xi}$$

Then

$$\mathbb{P}\left(x_m(t/\gamma) \le x\right) = \int \frac{\det(I - \lambda K)}{\prod_{k=0}^{m-1}(1 - \lambda\tau^{k-1})}\,\frac{d\lambda}{\lambda}$$

↑ k-1→k

The contour in the λ-plane encloses all of the singularities of the integrand.

Asymptotic analysis

We now transform the operator K so that we can perform a steepest descent analysis.

Recall that the generic behavior for the coalescence of two saddle points leads to the Airy function Ai(x)

George Airy

$$\xi \longrightarrow \frac{1-\tau\eta}{1-\eta}, \quad \tau = \frac{p}{q} < 1,$$

$$K(\xi,\xi') \longrightarrow K_2(\eta,\eta') = \frac{\varphi(\eta')}{\eta' - \tau\eta}$$

$$\varphi(\eta) = \left(\frac{1-\tau\eta}{1-\eta}\right)^x e^{\left[\frac{1}{1-\eta} - \frac{1}{1-\tau\eta}\right]\mathbf{t}}$$

Introduce: $K_1(\eta,\eta') = \dfrac{\varphi(\tau\eta)}{\eta' - \tau\eta}$

Two Preliminary Results

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Proposition:

Let Γ be any closed curve going around $\eta=1$ once counterclockwise with $\eta=1/\tau$ on the outside. Then the Fredholm determinant of $K(\xi,\xi')$ acting on C_R has the same Fredholm determinant as $K_1(\eta,\eta')-K_2(\eta,\eta')$ acting on Γ.

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Proposition:

Suppose the contour Γ is star-shaped with respect to $\eta=0$. Then the Fredholm determinant of K_1 acting on Γ is equal to

$$\prod_{k=0}^{\infty}(1-\lambda\tau^k)$$

Denote by R the resolvent kernel of K_1

Factor determinant:

$\det(I-\lambda K)=\det(I-\lambda K_1)\det(I+K_2(I+R))$

Set $\lambda=\tau^{-m}\mu$ so formula for distr. fn becomes

$$\int \prod_{k=0}^{\infty}(1-\mu\tau^k)\det\left(I+\tau^{-m}\mu K_2(I+R)\right)\ \frac{d\mu}{\mu}$$

μ runs over a circle of radius > τ

By a perturbative expansion of R, followed by a deformation of operators, we show

$$\begin{aligned}
\det\left(I + \lambda K_2(I+R)\right) &= \det\left(I + \mu J\right) \\
J(\eta,\eta') &= \int \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')} \frac{\zeta^m}{(\eta')^{m+1}} \frac{f(\mu,\zeta/\eta')}{\zeta-\eta}\, d\zeta \\
\varphi_\infty(\eta) &= (1-\eta)^{-x}\, e^{\frac{\eta t}{1-\eta}} \\
f(\mu,z) &= \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1-\tau^k\mu} z^k
\end{aligned}$$

The kernel J(η,η'), which acts on a circle centered at 0 with radius less than τ, is analyzed by the steepest descent method.

Note: m now appears inside the kernel!

Main Result

We set

$$\sigma = \frac{m}{t}, c_1 = -1+2\sqrt{\sigma}, c_2 = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}, \gamma = q-p$$

Theorem (TW). When $0 \leq p < q$ we have

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_m(t/\gamma) - c_1 t}{c_2 t^{1/3}} \leq s\right) = F_2(s)$$

Theorem also has a current fluctuation formulation

Total Current Fluctuations

I(x,t) = # of particles ≤ x at time t, x ≤0

Theorem.

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{I([-vt],t/\gamma)-a_1 t}{a_2 t^{1/3}} \le s\right) = 1 - F_2(-s)$$

where

$$0 \le v < 1,\ \ a_1 = \frac{1}{4}(1-v)^2,\ \ a_2 = 2^{-4/3}(1-v^2)^{2/3}$$

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- Balázs & Seppäläinen; Quastel & Valkó prove $t^{1/3}$ fluctuations for ASEP with Bernoulli product initial condition -- general probabilistic methods

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- Ultimately we want universality theorems not to rely upon integrable stucture of ASEP. For ⅓ exponent progress by Balázs, Seppäläinen, Quastel & Valkó.

Thanks to Anne Schilling & Doron Zeilberger for advice with the combinatorial identities