Dyson gas simulation of growing patterns: Geometry and integrability

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based on joint works with
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Abstract

We show how growth processes of Laplacian type can be simulated by statistical mechanics of 2D Coulomb charges in an external field (the Dyson gas), which may be thought of as eigenvalues of normal or complex random matrices. The growing cluster is represented by a domain where the mean density of the charges does not vanish in the large N limit, and the physical growth time is identified with a coupling constant of the external field. The Dyson gas picture applies both to Laplacian growth of smooth domains in the plane and to the growth of slit domains described by the Loewner equation. It also provides a key to integrable structure of the models and gives a unique way of their discretization or 'quantization' preserving integrability. From this point of view, we discuss growth models associated to the Toda lattice and KP integrable hierarchies.

Dyson gas = gas of 2D Coulomb particles (logarithmic potential) in an external field

Statistical weight:

$$e^{-\beta E} = \exp\left[\beta \sum_{i<j} \log |z_i - z_j|^2 - \beta \sum_i U(z_i)\right]$$

Partition function:

$$Z_N = \int \prod_{i<j}^{N} |z_i - z_j|^{2\beta} \prod_{i=1}^{N} d\mu(z_i, \bar{z}_i)$$

The measure:

- Smooth, with a support in the plane

$$d\mu(z,\bar{z}) = e^{\frac{1}{\hbar}W(z,\bar{z})}d^2z$$

- Singular, supported on lines

$$d\mu(z,\bar{z}) = e^{\frac{1}{\hbar}V(z)}\delta^{(2)}_{\Gamma}(z)d^2z$$

$$\int_{\mathbb{C}}(\ldots)d^2z_i \longrightarrow \int_{\Gamma}(\ldots)|dz_i|$$

Integrability

- $\beta = 1$
- $d\mu = d\mu_0 \exp\left(\mathcal{R}e \sum_{k\geq 1} t_k z^k\right)$

Then

$$\frac{1}{N!} Z_N = \tau_N(\{t_i\}; \{\bar{t}_i\})$$

is the **tau-function of the Toda lattice hierarchy**

Bilinear Hirota equations

$$\tau_n \, e^{\hbar \bar{D}(\bar{z}) + \hbar D(z)} \, \tau_n - e^{\hbar \bar{D}(\bar{z})} \tau_n \, e^{\hbar D(z)} \tau_n$$

$$= (z\bar{z})^{-1} \, \tau_{n-1} e^{\hbar \bar{D}(\bar{z}) + \hbar D(z)} \, \tau_{n+1}$$

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_{t_k}$$

(Toda lattice hierarchy)

The proof is based on the determinant representation

$$Z_N^{(\beta=1)} = \int \prod_{i<j}^{N} |z_i - z_j|^2 \prod_{i=1}^{N} d\mu(z_i, \bar{z}_i)$$

$$= N! \det_{N\times N}(C_{ij})$$

$$C_{ij} = \int z^i \bar{z}^j \, d\mu(z, \bar{z}), \qquad i, j = 0, 1, \ldots, N-1$$

which is a manifestation of the
free fermionic structure of the model

Random matrices:

At $\beta = 1$ z_i are eigenvalues of normal (or general complex) random matrices

$$Z_N^{(\beta=1)} \propto \int_{N\times N} \exp\left(\frac{1}{\hbar}\,\mathrm{tr}\, W(M, M^\dagger)\right)$$

(J.Ginibre, 1965; V.Girko, 1985;
L.-L.Chau and O.Zaboronsky, 1998)

Density of particles:

$$\rho(z) = \hbar \sum_{i=1}^{N} \delta^{(2)}(z - z_i)$$

The mean density: $\langle \rho(z) \rangle$

Normalization:

$$\int \langle \rho(z) \rangle \, d^2 z = N\hbar \; := t$$

Large *N* limit

$$N \to \infty, \quad \hbar \to 0, \quad N\hbar = t \ \text{ fixed}$$

$$Z_N = \int \exp\left(\frac{1}{\hbar^2} S(\{z_i\})\right) \prod_i d^2 z_i$$

The leading contribution comes from the maximum of *S* (the equilibrium configuration of the charges)

$$Z_N = \exp\left(\frac{F_0}{\hbar^2} + O(\hbar^{-1})\right)$$

The equilibrium condition:

$$\partial_{z_i} W(z_i, \bar{z}_i) = \hbar \sum_{j \neq i} \frac{1}{z_i - z_j}$$

or

$$\partial_z W(z, \bar{z}) = \int \frac{\rho(\xi) d^2\xi}{z - \xi}$$

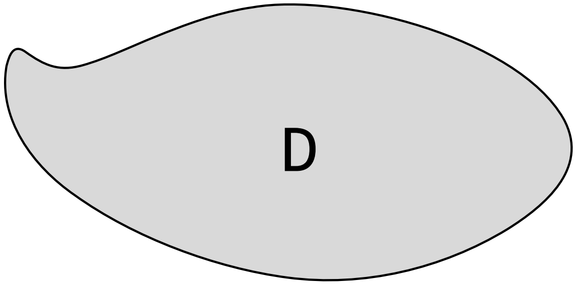
(for all z such that $\rho(z) \neq 0$)

Support of eigenvalues:

a domain D such that

$$\lim_{N\to\infty} \langle \rho(z) \rangle > 0 \quad \text{if} \quad z \in \mathrm{D}$$

and 0 otherwise



$$\lim_{N\to\infty} \langle \rho(z) \rangle = -\frac{\Delta W(z)}{4\pi}, \qquad z \in \mathrm{D}$$

The shape of D is a solution of the inverse potential problem

Example: $$W(z) = -|z|^2 + \mathcal{R}e \sum_{k \geq 0} t_k z^k$$

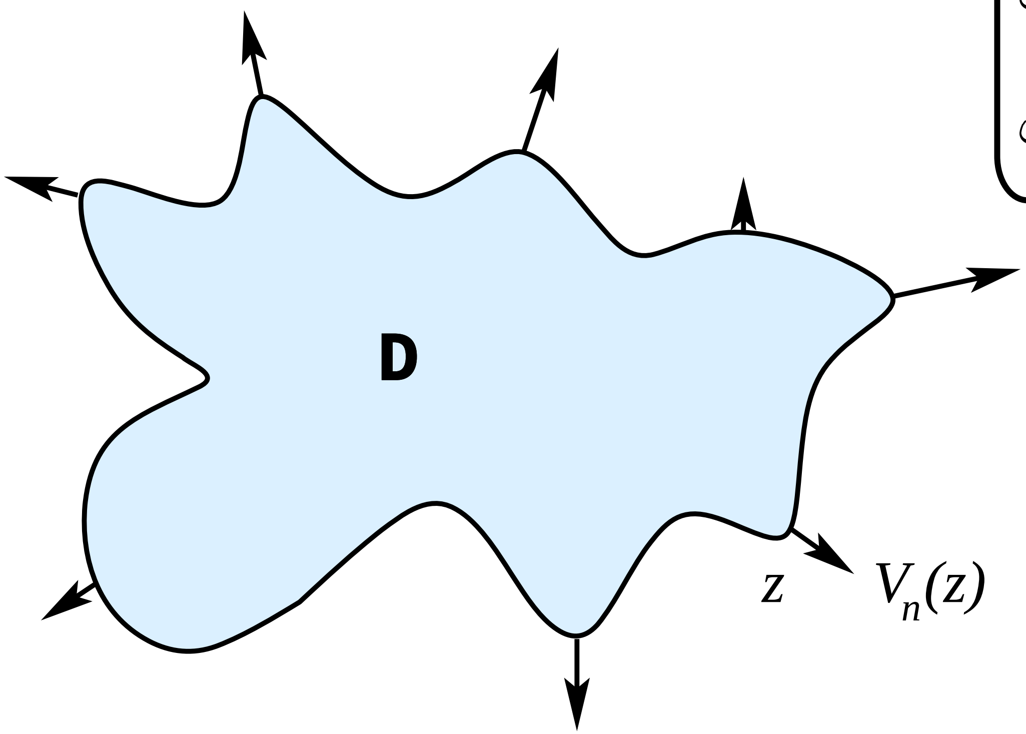
$$\langle \rho(z) \rangle = -1/\pi \quad z \in \mathrm{D}$$

$$\begin{cases} t_k = -\dfrac{1}{\pi k} \displaystyle\int_{\text{exterior of D}} z^{-k} d^2 z \\ t = N\hbar = \dfrac{\text{area (D)}}{\pi} \end{cases}$$

Growth of D as *N* increases = Laplacian growth

'Time': $t = N\hbar$

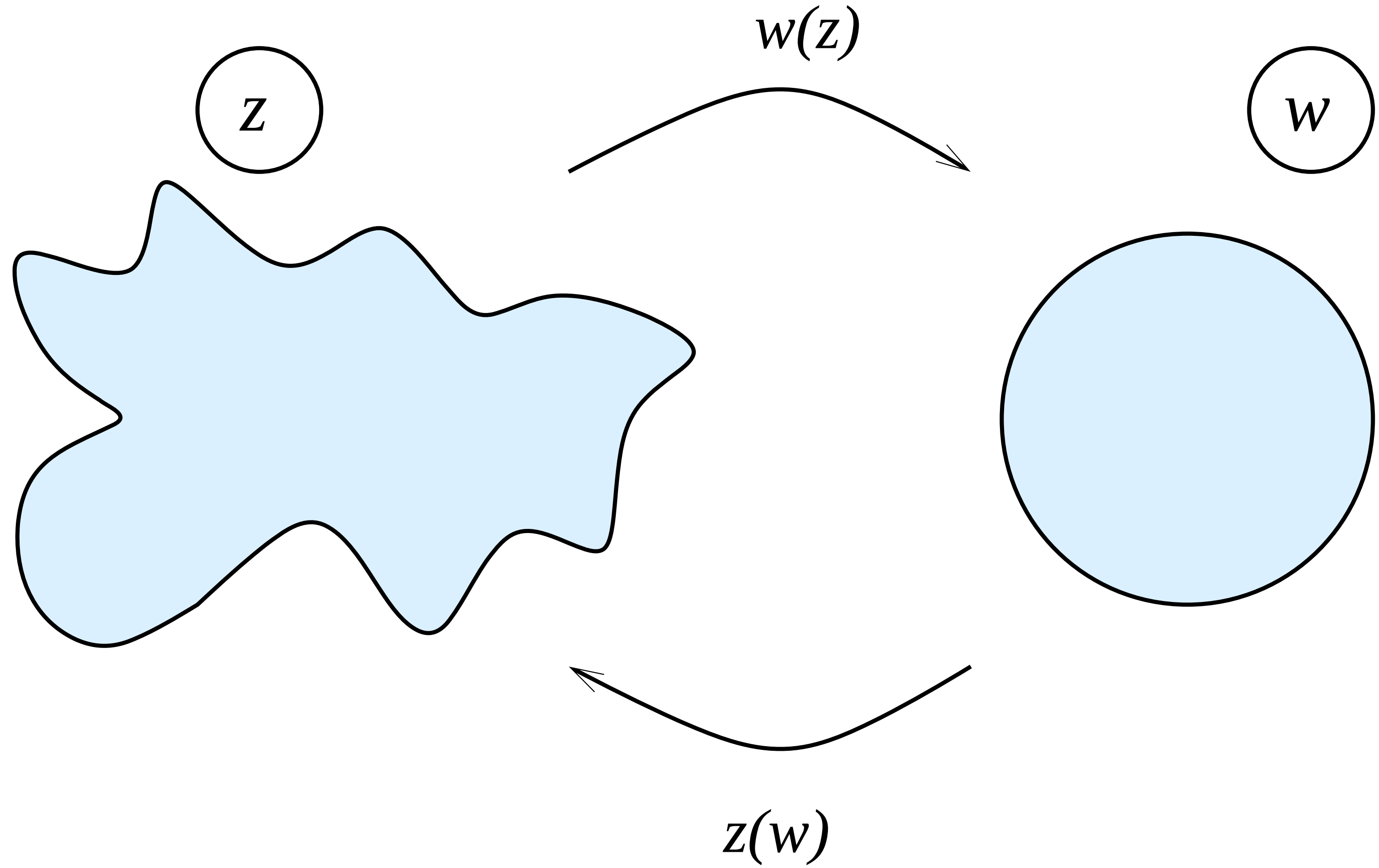
Laplacian growth
$$\begin{cases} V_n(z) = \partial_n \varphi(z) \quad \text{(Darcy law)} \\ \Delta\varphi(z) = 0 \ \text{outside D} \\ \varphi(z) = 0 \ \text{on the boundary} \\ \varphi(z) \to -\log|z| \ \text{as} \ z \to \infty \end{cases}$$



$$\varphi(z) = \log|w(z)|$$

w(z) is conformal map from exterior of D *to exterior of unit circle*

Conformal maps:



$$z(w) = rw + u_0 + \frac{u_1}{w} + \frac{u_2}{w^2} + \ldots$$

Integrability in the large *N* limit: dispersionless Toda hierarchy (d-Toda)

‘Slow’ times: $t_k \longrightarrow \dfrac{t_k}{\hbar}\,, \qquad \hbar \longrightarrow 0$

$$e^{\hbar\partial} \longrightarrow w\,, \qquad L\left(e^{\hbar\partial}\right) \longrightarrow z(w)$$

$$\partial_{t_k} L = [A_k, L]$$

$$\Big\downarrow$$

$$\partial_{t_k} z(w) = \{A_k, z(w)\}$$

Lax equations for the d-Toda hierarchy

Lax equations:

$$\frac{\partial z(w)}{\partial t_k} = \{A_k(w),\, z(w)\}$$

$$\frac{\partial z(w)}{\partial \bar{t}_k} = -\{\bar{A}_k(w^{-1}),\, z(w)\}$$

$$A_k(w) = (z^k(w))_+ + \frac{1}{2}(z^k(w))_0\,, \qquad k \geq 1$$

Notation (Poisson bracket):

$$\{f, g\} = w\left(\frac{\partial f}{\partial w}\frac{\partial g}{\partial t} - \frac{\partial g}{\partial w}\frac{\partial f}{\partial t}\right)$$

Compatibility of the Lax equations yields the **d-Toda hierarchy**, an infinite system of PDE's in infinitely many variables t_i and infinitely many unknown functions r, u_i.

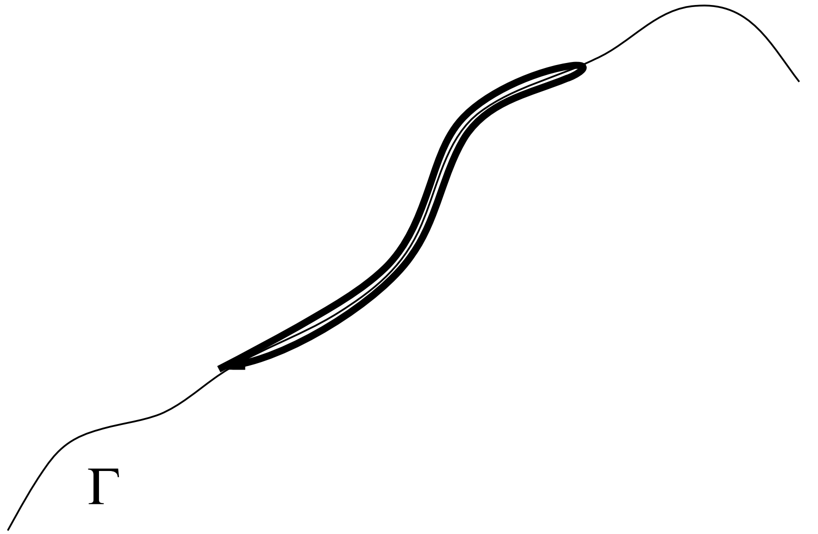
The simplest equation:

$$\frac{\partial^2 \log r^2}{\partial t_1 \partial \bar{t}_1} = \frac{\partial^2 r^2}{\partial t^2}$$

The "dispersionless τ-function":

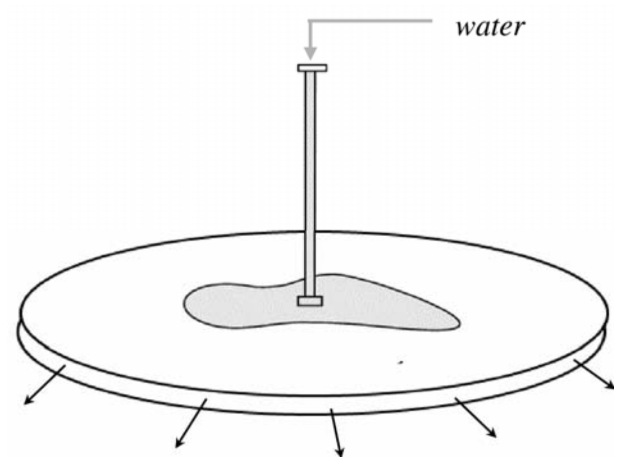
$$F_0 = \lim_{\hbar \to 0} \left(\hbar^2 \log Z_N \right) = -\frac{1}{\pi^2} \int_{\mathsf{D}} \int_{\mathsf{D}} \log \left| \frac{z - \zeta}{z \zeta} \right| d^2 z d^2 \zeta$$

If the measure $d\mu$ in the integral over z_i is supported on a curve, the 'particles' fill an arc of the curve.



The function *z(w)* is a conformal map to a slit domain

Laplacian growth in the whole plane and 2D Toda hierarchy

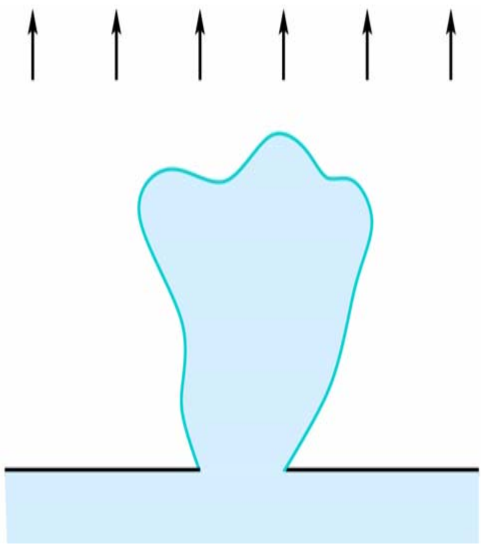


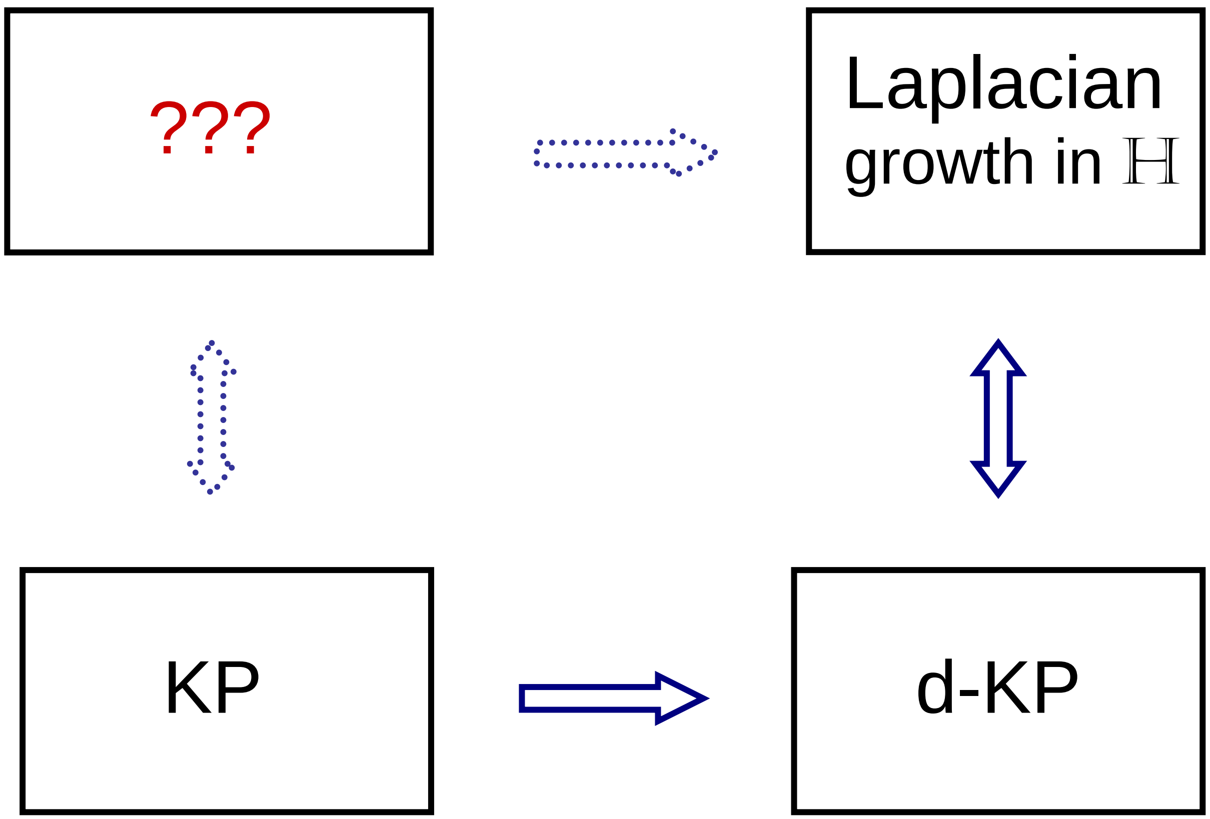
Dyson gas $\xrightarrow{N \to \infty}$ Laplacian growth in $\mathbb{C}$

$\Updownarrow$ $\Updownarrow$

Toda $\Longrightarrow$ d-Toda

Laplacian growth in the upper half-plane and KP hierarchy





The d-KP hierarchy

Consider a Laurent series

$$z(p) = p - \frac{u}{p} + \sum_{k=2}^{\infty} u_k p^{-k}$$

where u, u_i depend on real parameters

$$T_1, T_2, T_3, \ldots$$

Lax equations:

$$\frac{\partial z(p)}{\partial T_k} = \{B_k(p),\, z(p)\}$$

where

$$B_k(p) = (z^k(p))_{\geq 0}\,, \quad k \geq 1$$

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial T_1} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial T_1} \quad \text{(Poisson bracket)}$$

This gives an infinite system of PDE's (the **d-KP hierarchy**) in infinitely many variables T_i and infinitely many unknown functions

$$u_i = u_i(T_1, T_2, T_3, \ldots)$$

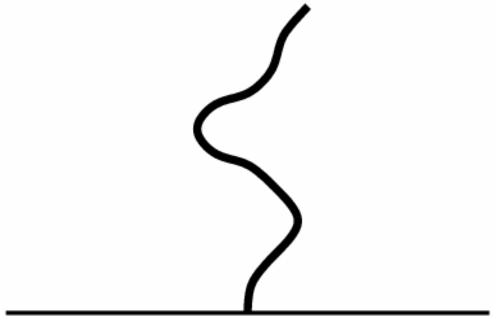
The simplest equation (Khohlov-Zabolotskaya equation):

$$3\frac{\partial^2 u}{\partial T_2^2} + 4\frac{\partial}{\partial T_1}\left(\frac{\partial u}{\partial T_3} + 3u\frac{\partial u}{\partial T_1}\right) = 0$$

d-KP ⟹ Conformal maps ???

A partial answer (J.Gibbons and S.Tsarev, 1999):

- For a special class of solutions to d-KP ("finite dimensional reductions"), $z(p)$ maps the upper half plane to the complement of a growing slit.



Loewner equation

Let $z(p)$ be a solution to Lax equations for d-KP depending on $\{T_i\}$ through only one function $u = u(T_1, T_2, T_3, \ldots)$:

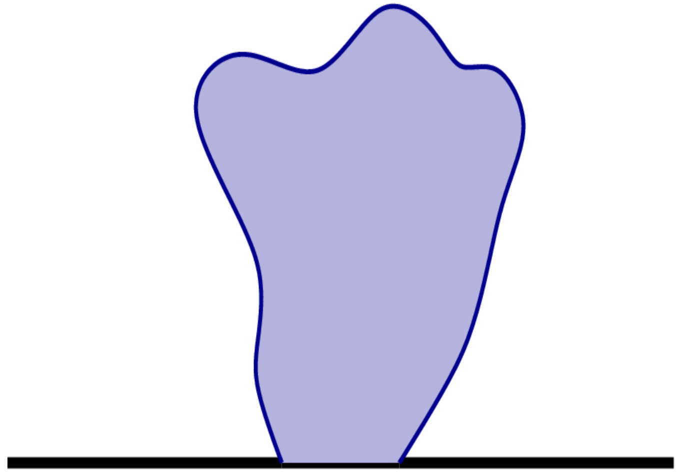
$$z(p; \{T_i\}) = z(p; u(\{T_i\})$$

Then $z(p; u)$ obeys the (chordal) Loewner equation

$$\frac{\partial z(p)}{\partial u} = \frac{1}{p - \xi(u)} \frac{\partial z(p)}{\partial p}$$

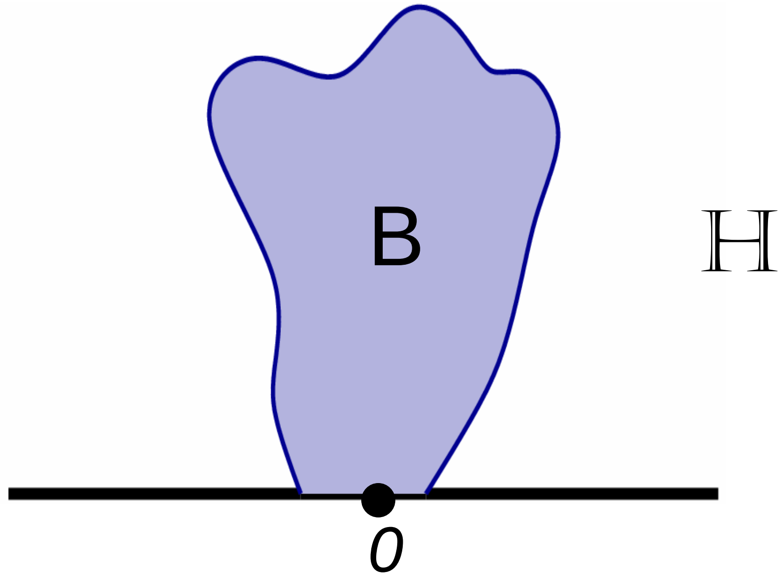
which serves as a consistency condition for the reduction of the hierarchy.

More general solutions to d-KP describe growth of **‘fat slits’**



They extend the Gibbons-Tsarev picture and serve as a ‘d-KP version’ of Laplacian growth

Harmonic moments of a fat slit

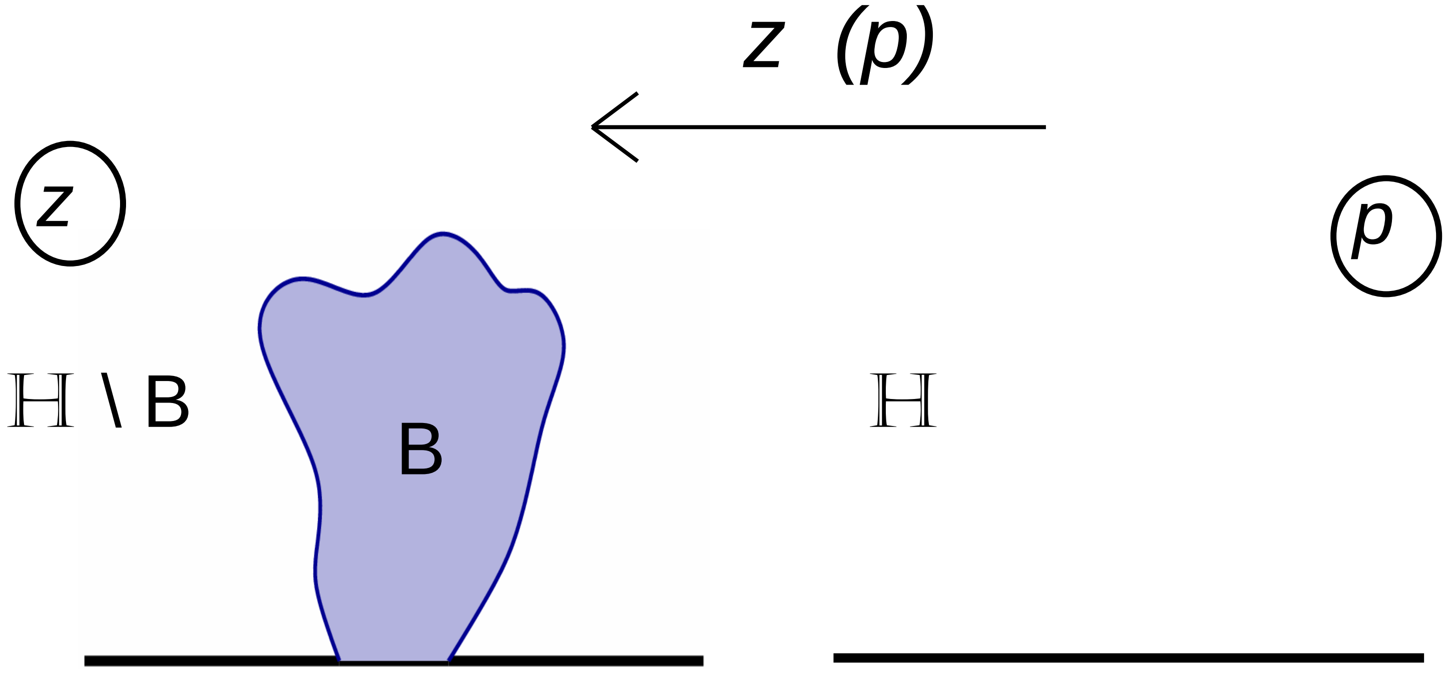


$$\begin{cases} T_k = \dfrac{2}{\pi k}\mathcal{I}m\displaystyle\int_{\mathbb{H}\setminus \mathsf{B}} z^{-k} d^2 z\,, \quad k \geq 2, \\ T_1 = -\dfrac{2}{\pi}\mathcal{I}m\displaystyle\int_{\mathsf{B}} z^{-1} d^2 z \end{cases}$$

Conformal map $z(p) = p - \frac{u}{z} + O(z^{-2})$

obeys the Lax equations of the d-KP hierarchy

$$\frac{\partial z(p)}{\partial T_k} = \{(z^k(p))_{\geq 0},\, z(p)\}$$

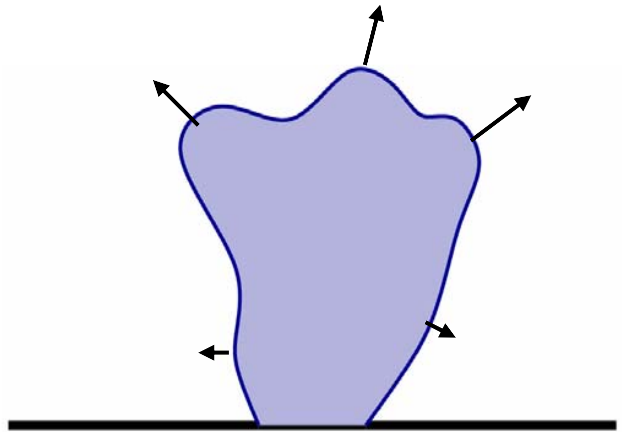


Laplacian growth in the upper half-plane

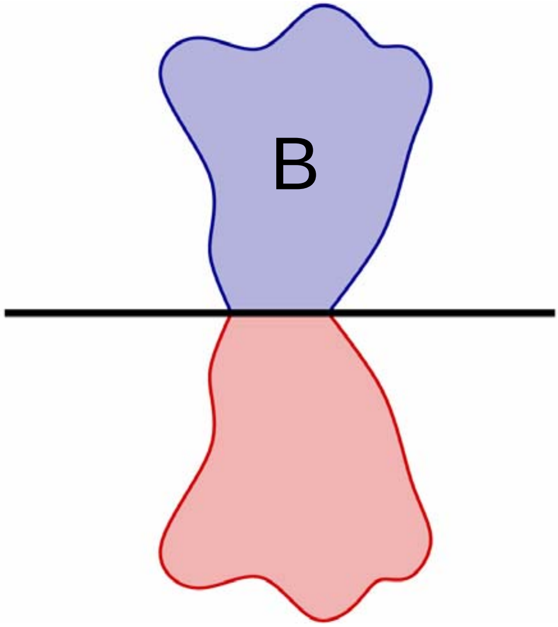
Let all T_k be constant except for $T_1 = T$. Then the boundary of a fat slit moves with normal velocity

$$V_n(z) = \frac{1}{2}\partial_n \mathcal{I}m\, p(z)\,, \qquad z \text{ on the boundary}$$

where $p(z)$ is the inverse conformal map to $z(p)$.

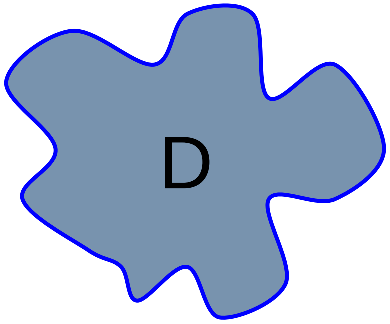


Dispersionless tau-function



$$F^{\mathrm{KP}} = -\frac{1}{\pi^2}\int_{\mathrm{B}}\int_{\mathrm{B}} \log\left|\frac{z-\zeta}{z-\bar{\zeta}}\right| d^2z\, d^2\zeta$$

Compare with



$$F^{\mathrm{Toda}} = -\frac{1}{\pi^2}\int_{\mathrm{D}}\int_{\mathrm{D}} \log\left|\frac{z-\zeta}{z\zeta}\right| d^2z\, d^2\zeta$$

Dispersive version

The grand canonical ensemble:

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i<j}^{N} \left| \frac{z_i - z_j}{z_i - \bar{z}_j} \right|^2 \prod_{l=1}^{N} d\mu(z_l, \bar{z}_l)$$

$$d\mu(z, \bar{z}) = d\mu_0(z, \bar{z}) \exp \left(\frac{1}{\hbar} \sum_{k \geq 1} T_k (z^k - \bar{z}^k) \right)$$

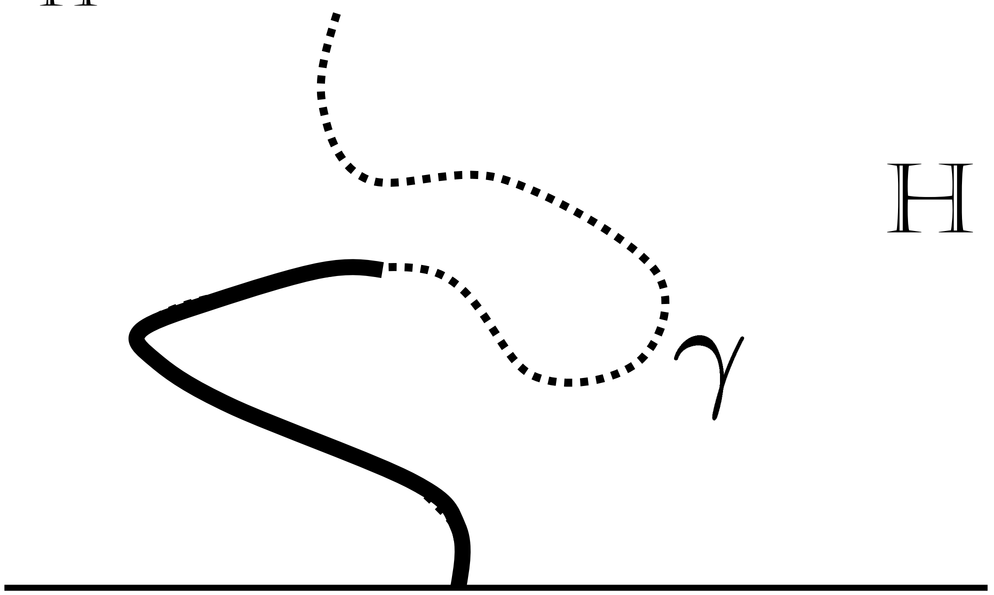
(for example, $d\mu_0 = e^{-y^2/\hbar}$)

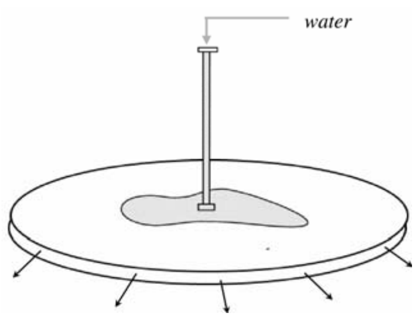
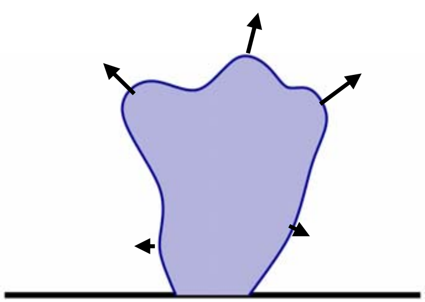
$$\boxed{\mathcal{Z} = \tau^{\mathrm{KP}}(\{T_k\})}$$

Limit to slits:

$$d\mu_0(z,\bar{z}) \to \delta_\gamma(z) d^2 z$$

$$\int_{\mathbb{H}} (\ldots) d^2 z \to \int_\gamma (\ldots) |dz|$$



Growth problem	Integrable hierarchy	Dyson gas realization
 LG in the plane	d-Toda $t = t_0,$ $\{t_k\}, \{\bar{t}_k\}, \; k \geq 1$	Canonical ensemble $N \to \infty, \hbar \to 0$ $N\hbar = t$
 LG in the half-plane	d-KP $T = T_1,$ $\{T_k\}, \; k \geq 2$	Grand canonical ensemble $\hbar \to 0$